

University of Alberta

Electric Dipole Moment of the Electron

by

John Paul Archambault



A thesis submitted to the Faculty of Graduate Studies and Research in partial
fulfillment of the requirements for the degree of Master of Science

Department of Physics

Edmonton, Alberta

Spring 2003



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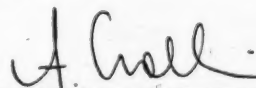
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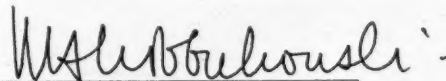
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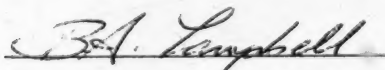
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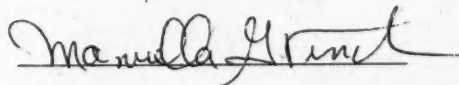
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Abstract

The electron electric dipole moment (EDM) is investigated as a manifestation of CP violation in the lepton sector. It is shown that massive Majorana neutrinos can induce an electron EDM at the two-loop level. After developing the necessary background formalism, a detailed calculation is presented. Using the method of asymptotic expansions, analytic expressions are obtained for the electron EDM as a function of arbitrary masses of the Majorana neutrinos. The results indicate that the electron EDM will be sufficiently small, for plausible values of neutrino masses, so as to be experimentally unobservable in the foreseeable future.

Acknowledgements

First, I would like to thank my supervisor, Dr. Andrzej Czarnecki, for his guidance throughout this research project. I am also grateful to Ian Blokland who was a tremendous help in learning the techniques needed for the project. To the rest of the guys, thank you for keeping me sane with your discussions on anything and everything.

I would like thank Erin for helping me through many tough times throughout my studies. I would also like to thank all of my family and friends back home. I always look forward to a phone call or a visit.

And of course, I would like to thank Mom, Dad and Mike for everything. I couldn't have done this without them.

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Chapter 1

Introduction

The study of electric dipole moments (EDMs) offers a look into exciting physical ideas. A non-zero electron EDM would provide an indication of CP violation in the lepton sector and thus would have many implications in high-energy physics, such as a possible explanation of the baryon asymmetry of the universe. CP violation has only been observed in the quark sector thus far but it is expected to exist in the lepton sector as well.

The study of the EDM of the electron probes physics beyond the Standard Model. Within the Standard Model, the Cabibbo-Kobayashi-Maskawa (CKM) matrix is a source for CP violation and accounts for the only experimentally verified cases of CP violation: the B -meson system [1, 2] and the K -meson system [3]. Another example of CP violation via the CKM model would be the EDM of the neutron, however the CKM model predicts very small values for the neutron EDM. Standard Model estimates of the neutron EDM are in the neighbourhood of 10^{-31} e cm [4–6]. The current experimental limit on the neutron EDM is $d_n < 0.63 \times 10^{-25}$ e cm [7].

The CKM model can also induce an EDM in the charged leptons by closed quark loops, however this effect only begins to contribute at the four-loop level

and is therefore very heavily suppressed. Standard Model estimates for the electron EDM are about 10^{-50} e cm [8, 9], about 20 orders of magnitude smaller than the current experimental limit [10].

Another possible source for CP violation is mixing within the lepton sector. For example, with the existence of Majorana neutrinos, an EDM can be induced in the electron at the two-loop level. Although massive Majorana neutrinos are not present in the Standard Model, they often arise in its various extensions. They are interesting because they can lead to novel effects at a much lower order in perturbation theory than the Standard Model would otherwise allow.

Many atomic physics experiments have been performed to measure the electron EDM with the most recent result [10] placing an upper limit of $|d_e| \leq 1.6 \times 10^{-27}$ e cm. Future experiments using solid state systems [11] are expected to have a sensitivity of 10^{-32} e cm.

A numerical calculation of the two-loop effect of Majorana neutrinos on the EDM of the electron was first performed by Ng and Ng [12]. In this thesis, the method of asymptotic expansions is employed to obtain an analytic expression for the electron EDM.

Before discussing the calculation of the electric dipole moment of the electron, Chapter 2 will explore a number of important background topics. The first of these are the discrete symmetries of charge conjugation (C), parity (P) and time reversal (T). Exploring these symmetries naturally leads to the idea of CP violation and the kaon system will be used as an illustrative example. Next, a study of massive neutrinos introduces the concepts of neutrino mixing as well as Majorana particles and how they affect CP violation in the lepton sector. It will be shown how the seesaw mechanism generates very heavy and very light Majorana particles that may contribute to the EDM of the electron. This mechanism will allow for calculations of the EDM in three different limits of the Majorana

neutrino mass. The case where the Majorana mass is small ($m_{i,j} \ll M_W$) will build on the work of [12]. Then, further calculations are performed to find the behaviour of the EDM as the Majorana mass increases toward and beyond M_W . The first parts of these calculations will be arranged at the end of Chapter 2. In particular, the form of the electric dipole moment will be given and the technique that is used to extract the relevant quantity from the Feynman diagrams will be illustrated.

In Chapter 3 the methods of calculation will be thoroughly examined. The starting point will be the introduction of the diagrams that contribute to the EDM of the electron and the establishment of Feynman rules for dealing with Majorana particles. A brief introduction on the method of asymptotic expansions is then given and a detailed description of each of the three limits will be presented and used as examples of the method.

In Chapter 4 the results of the calculations are presented. The effect of the light Majorana neutrino is compared to the results in [12] and to current and future experimental limits of the electron EDM. The behaviour of the EDM is also provided for the effect of the heavy Majorana particle as its mass increases toward and beyond M_W .

Finally, Chapter 5 summarizes all the topics and results and provides an outlook for future calculations.

Chapter 2

Theory

2.1 CPT Properties

Before the idea of CP violation is explored, the individual discrete symmetries of parity (P), charge conjugation (C) and time reversal (T) will be introduced.

2.1.1 Parity

A parity transformation is one which essentially brings a process to its mirror image. More specifically, the position vector, $\mathbf{r}$, is transformed into $-\mathbf{r}$ under a parity transformation, P. As a result, the linear momentum of a particle also changes sign. However, the angular momentum of the particle is unaffected by a parity transformation:

$$\begin{aligned}\mathbf{p} &= m \frac{d\mathbf{r}}{dt} \xrightarrow{P} m \frac{d(-\mathbf{r})}{dt} = -\mathbf{p} \\ \mathbf{L} &= \mathbf{r} \times \mathbf{p} \xrightarrow{P} (-\mathbf{r}) \times (-\mathbf{p}) = \mathbf{L}\end{aligned}\quad (2.1)$$

For consistency, all angular momentum, including spin, is unchanged by a parity transformation. Thus,

$$P |a(\mathbf{p}, s)\rangle = |a(-\mathbf{p}, s)\rangle \quad (2.2)$$

where $\mathbf{p}$ is the momentum of the state a with spin s . It is then said that if a system is a right-handed system, P transforms it into an upside-down and backward-moving left-handed system.

If P is applied twice, the original system is obtained:

$$P^2 = 1 \quad (2.3)$$

The eigenvalues for P are thus ± 1 . Hadrons are eigenstates of parity and can be classified by their eigenvalue just as they are classified by their spin, charge, isospin and strangeness.

Parity is a multiplicative quantum number; that is, the parity of a composite system in its ground state is the product of the parities of its components. For states of angular momentum l , there is an extra factor of $(-1)^l$ which results from the spherical harmonics that describe the angular dependence of the wavefunction.

Historically, it was taken for granted that the laws of physics were invariant under parity transformations. There was ample evidence for parity invariance in the strong and electromagnetic interactions, but in 1956, Lee and Yang [13] put forward a parity-violating model of weak interactions.

To test the idea of parity violation in the weak interaction, an experiment was proposed, and performed in 1957 [14]. In the experiment, radioactive cobalt-60 nuclei were carefully aligned so as to have their spins pointing in the z -direction. Since cobalt-60 undergoes β -decay ($n \rightarrow pe^- \bar{\nu}_e$), the direction of the emitted electrons can be measured. It was found that the majority of the electrons were emitted in the direction of the spin; that is, the electrons were emitted in the $+z$ -direction. This simple experiment was enough to demonstrate parity violation in the weak interaction. To see this, imagine the mirror image of the process. The image nucleus rotates in the opposite direction, its spin points downward and yet the electrons are still emitted in the same direction. In particular, the

electrons are now emitted opposite to the direction of the spin. In order for this process to possess parity invariance, the electrons would have to be equally likely to be emitted in both directions.

It is known that in the weak interaction, parity is "maximally" violated. This means that the ν - e - W coupling has both a vector and an axial vector term that contribute with the same strength.

2.1.2 Charge Conjugation

In particle physics, an operation which reverses the sign of a particle's charge is referred to as charge conjugation. Denoted by C , charge conjugation is defined to transform a fermion with a given spin into an antifermion with the same spin:

$$C |a(\mathbf{p}, s)\rangle = |b(\mathbf{p}, s)\rangle \quad (2.4)$$

Charge conjugation actually changes the sign of all internal quantum numbers, including charge, baryon number, lepton number, strangeness, charm, beauty and truth, while leaving mass, energy, momentum and spin unchanged.

As in the case with parity, acting twice with the charge conjugation operator on a system returns it to the original system up to a phase. However, most particles in nature are not eigenstates of C : if $|a(\mathbf{p}, s)\rangle$ is an eigenstate of C , it follows that

$$C |a(\mathbf{p}, s)\rangle = |b(\mathbf{p}, s)\rangle = \eta_C |a(\mathbf{p}, s)\rangle \quad (2.5)$$

so that $|a(\mathbf{p}, s)\rangle$ and $|b(\mathbf{p}, s)\rangle$ differ at most by a phase, η_C . This means that they represent the same physical state, thus, only those particles that are their own antiparticles can be eigenstates of C . This becomes important when dealing with Majorana neutrinos as will become evident in the following sections.

Because the photon is the quantum of the electromagnetic field (which changes sign under C), it makes sense that the photon's charge conjugation

number is (-1) . It can be shown [15] that a system consisting of a spin-1/2 particle and its antiparticle in a configuration with orbital angular momentum l and total spin s constitutes an eigenstate of C , with eigenvalue $(-1)^{l+s}$.

Charge conjugation is also a multiplicative quantum number and, like parity, it is conserved in strong and electromagnetic interactions. As an example, the neutral pion will decay into two photons,

$$\pi^0 \rightarrow \gamma\gamma \quad (2.6)$$

but never three photons since the π^0 is a pseudoscalar ($l = s = 0$, $P = -1$). In general, the π^0 will decay into n photons only if n is an even number.

On the other hand, charge conjugation is not a symmetry of the weak interaction. When applied to a neutrino (left-handed), C produces a left-handed anti-neutrino which is absent in the Standard Model. Thus the charge conjugation of any process involving neutrinos is physically impossible in the Standard Model.

2.1.3 Time Reversal

The transformation corresponding to time reversal, T , is the reversal of the clock, $t \rightarrow -t$, while the spatial coordinates remain unchanged. As a result, both the linear momentum and angular momentum of a particle are reversed.

$$\begin{aligned} \mathbf{p} &= m \frac{d\mathbf{r}}{dt} \xrightarrow{T} m \frac{d\mathbf{r}}{d(-t)} = -\mathbf{p} \\ \mathbf{L} &= \mathbf{r} \times \mathbf{p} \xrightarrow{T} \mathbf{r} \times (-\mathbf{p}) = -\mathbf{L} \end{aligned} \quad (2.7)$$

Again, for consistency, all angular momentum reverses sign under time reversal. The major difference between time reversal and the other discrete transformations is that it is an antiunitary transformation. Thus, the time transformation operator acting on a wavefunction has the property

$$T\psi(\mathbf{x}, t) = \psi^*(\mathbf{x}, -t) \quad (2.8)$$

This property is best illustrated by looking at the free Schrödinger equation with the plane-wave solution $\psi(\mathbf{x}, t) = e^{i(Et - \mathbf{p} \cdot \mathbf{x})}$. It is easily seen that the wavefunction is a solution to the Schrödinger equation if $E = p^2/2m$. However, $\psi(\mathbf{x}, -t) = e^{-i(Et + \mathbf{p} \cdot \mathbf{x})}$ is only a solution if $E = -p^2/2m$, which is not acceptable for a physical particle. The solution to this problem is to insist that the time reversal operator act as given in Eq. (2.8). The new wavefunction, $\psi^*(\mathbf{x}, -t) = e^{i(Et + \mathbf{p} \cdot \mathbf{x})}$, is once again a solution with energy $E = p^2/2m$.

An important aspect of time reversal is that it is intimately connected to charge conjugation and parity transformations through the TCP theorem. The TCP theorem states that the combined operations of charge conjugation, parity transformations and time reversal (in any order) is an exact symmetry of all interactions in any Lorentz invariant quantum field theory. As a consequence, any violation of CP must be compensated by a violation in T.

2.1.4 CP Violation in the Quark Sector

It has been shown that the weak interaction is not invariant under parity transformations or charge conjugation. However, it seemed that an invariance could be restored after combining the two operations. This is illustrated in the decay

$$\pi^+ \rightarrow \mu^+ \nu_\mu \quad (2.9)$$

which does not conserve parity due to the fact that the antimuon is always emitted left-handed. The charge conjugate decay would be

$$\pi^- \rightarrow \mu^- \bar{\nu}_\mu \quad (2.10)$$

with a left-handed muon. However, when this decay is observed, the muon is always emitted right-handed. The operation of CP on the original π^+ decay will produce the π^- decay with a right-handed muon. This is exactly what is

needed and the symmetry seems to be restored in the lepton sector of the weak interaction. This is not the case for the quark sector.

The effect of CP invariance on neutral K mesons was first explored by Gell-Mann and Pais [16]. They reported that the K^0 , having strangeness of $(+1)$ could turn into its antiparticle $\bar{K}^0$ with strangeness of (-1) through second order weak interactions shown in Figure 2.1.

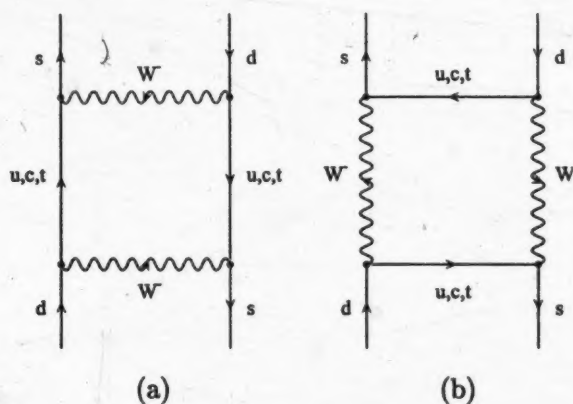


Figure 2.1: Feynman diagrams contributing to $K^0 - \bar{K}^0$ mixing

This means that the particles observed in the laboratory are not necessarily K^0 and $\bar{K}^0$, but some linear combination of the two. In particular, eigenstates of CP can be formed as follows: since the K^0 's are pseudoscalars ($l = s = 0$)

$$P|K^0\rangle = -|K^0\rangle, \quad P|\bar{K}^0\rangle = -|\bar{K}^0\rangle \quad (2.11)$$

and since $C|a(\mathbf{p}, s)\rangle = |b(\mathbf{p}, s)\rangle$,

$$C|K^0\rangle = |\bar{K}^0\rangle, \quad C|\bar{K}^0\rangle = |K^0\rangle \quad (2.12)$$

So

$$CP|K^0\rangle = -|\bar{K}^0\rangle, \quad CP|\bar{K}^0\rangle = -|K^0\rangle \quad (2.13)$$

This gives the eigenstates of CP:

$$\begin{aligned} |K_1\rangle &= \frac{1}{\sqrt{2}} (|K^0\rangle - |\bar{K}^0\rangle) \\ |K_2\rangle &= \frac{1}{\sqrt{2}} (|K^0\rangle + |\bar{K}^0\rangle) \end{aligned} \quad (2.14)$$

with

$$\text{CP}|K_1\rangle = |K_1\rangle, \quad \text{CP}|K_2\rangle = -|K_2\rangle \quad (2.15)$$

With the assumption that CP is conserved in weak interactions, K_1 can only decay into a state with CP = +1 and K_2 can only decay into a state with CP = -1. The most typical decay of a neutral kaon is into either 2π or 3π . Since the π 's consist of fermion-antifermion pairs with $l = 0$, each will have a parity of (-1). The 2π state will then have a parity of (+1) and the 3π state will have a parity of (-1). Thus, the K_1 can only decay into 2π and the K_2 can only decay into 3π . Since the energy released is greater in the 2π decay, this reaction occurs faster. Thus, if a beam consists of

$$|K^0\rangle = \frac{1}{\sqrt{2}} (|K_1\rangle + |K_2\rangle) \quad (2.16)$$

we should expect that after some sufficiently long time there will be a beam consisting of only K_2 mesons. That is, the K_1 mesons will decay earlier, producing many 2π events while the 3π events will occur later.

The lifetimes of the particles were found to be [17]

$$\begin{aligned} \tau_1 &= 0.89 \times 10^{-10} \text{ s} \\ \tau_2 &= 5.2 \times 10^{-8} \text{ s} \end{aligned}$$

with the K_1 -lifetime being much shorter than that of K_2 , as expected.

The neutral kaon system provides an ideal experimental system to test for CP violation. If the beam is long enough, there should be a region where only 3π events should occur. If any 2π events are observed, it will show that CP is

violated. In 1964, 45 2π events were observed in a total of 22,700 decays [3]. Although the number was small, it was nevertheless evidence of CP violation. As a result, the long-lived neutral kaon is not a perfect eigenstate of CP; rather it contains a small admixture of K_1 :

$$|K_{long}\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}} (|K_2\rangle + \epsilon|K_1\rangle) \quad (2.17)$$

where ϵ is a measure of the extent of the CP violation. CP violation is a small effect and was only found in the kaon system until recently, where it was observed in B -meson systems [1, 2].

The Standard Model incorporates CP violation with a complex phase in the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix for quarks:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \quad (2.18)$$

where $c_i = \cos(\theta_i)$, $s_i = \sin(\theta_i)$ and δ is the CP-violating phase. In the quark sector, at least three generations of quarks are needed in order to introduce a CP-violating phase.

The importance of the complex phase can be seen in considering time reversal. As shown earlier, time reversal is an antiunitary operator, which means that complex conjugation is part of the operator. This means that the factor of $e^{i\delta}$ that will appear in the weak interaction coupling constants will be transformed into $e^{-i\delta}$ by the time reversal operator and will have noticeable effects. In order for a theory to be invariant under CPT transformations, there must be a violation in CP to compensate for the violation in T.

2.2 Neutrino Physics

2.2.1 Introduction to Neutrinos

The first mention of neutrino mass came from Pauli, who proposed a particle with a mass less than that of an electron. A method to detect the mass of the neutrino was first set by Fermi [18] and Perrin [19]. The method was based on the high-energy part of the tritium β -decay spectrum

$${}^3\text{H} \rightarrow {}^3\text{He} + e^- + \bar{\nu}_e. \quad (2.19)$$

The first experiments showed no effects of a neutrino mass. Throughout the years, the upper bound on the mass fell to about 100 eV. With this limit in place, the theory of the neutrino as well as the electroweak theory were based on the assumption that the neutrino was a massless particle.

The first serious case in which the neutrino was thought to have mass was proposed by Pontecorvo [20]. Pontecorvo argued that, unlike the photon, whose masslessness is protected by the gauge invariance of QED, there was no known symmetry associated with the neutrino.

In this section, some consequences of neutrino mass are introduced, including neutrino mixing and Majorana neutrinos. Emphasis will be placed on Majorana neutrinos and their properties.

2.2.2 Neutrino Mixing

If neutrinos have mass, their mass matrix will in general be non-diagonal and complex, as in the case of the quarks. The mass matrix must then be diagonalized by unitary rotations and thus the mass eigenstates might differ from the gauge eigenstates. The gauge eigenstates can be written as

$$\nu_\alpha = \sum_i U_{\alpha i} \nu_i \quad (2.20)$$

where $\nu_i \equiv \nu_1, \nu_2, \nu_3$ are the mass eigenstates with eigenvalues m_1, m_2, m_3 . U is a unitary matrix which can be parametrised by the Maki-Nakagawa-Sakata (MNS) matrix which is the neutrino sector analogue of the CKM matrix for quark mixing of Eq. (2.18).

Neutrino mixing leads to the phenomenon of neutrino oscillations, whereby a neutrino (produced through weak interaction decays and corresponding to a definite flavour) spontaneously changes into a neutrino of a different flavour while travelling in vacuum (or matter). Neutrino oscillations perhaps provide the best opportunity to detect neutrino mass as many experiments, including those at the Sudbury Neutrino Observatory (SNO) are sensitive to this phenomenon. For example, the SNO experiment studies the flux ϕ of high energy neutrinos from the decay of ${}^8\text{B}$ in the sun. The solar neutrinos are detected by the reactions

$$\nu + d \rightarrow e^- + p + p \quad (2.21)$$

$$\nu + d \rightarrow \nu + p + n \quad (2.22)$$

$$\nu + e \rightarrow \nu + e \quad (2.23)$$

The first reaction is only triggered by ν_e while the other two reactions can be initialized by any of the active flavours of neutrinos. The results of the measurements show that neutrino fluxes are [21]

$$\phi(\nu_e) = (1.76 \pm 0.10) \times 10^6 \text{ cm}^{-2}\text{s}^{-1} \quad (2.24)$$

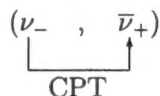
$$\phi(\nu_{\mu,\tau}) = (3.14^{+0.66}_{-0.64}) \times 10^6 \text{ cm}^{-2}\text{s}^{-1} \quad (2.25)$$

Since the nuclear processes that power the sun produce only ν_e and not ν_μ or ν_τ , these results are clear evidence that neutrinos do indeed change flavour.

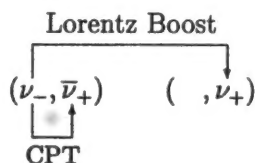
2.2.3 Dirac and Majorana Particles

If the neutrino is a massive particle, a distinction can be made between Dirac particles and Majorana particles. Suppose that there exists a massive neutrino

with negative helicity, ν_- . In a CPT-invariant quantum field theory, there exists its CPT mirror image. That is, there exists an antineutrino with positive helicity, $\bar{\nu}_+$.

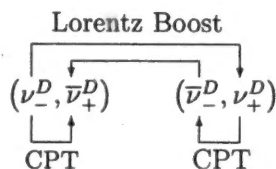


If the neutrino has a mass, a Lorentz boost can be made to a frame travelling faster than the neutrino. In this frame, the neutrino is travelling in the opposite direction but with the same spin, therefore there has been a transformation between a negative helicity state to a positive helicity state via a Lorentz boost:



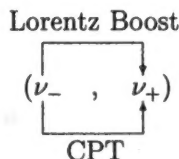
A question arises from this demonstration and the fact that the neutrino has no "charges": Are the two states with positive helicity the same particle?

If the answer is *no*, then ν_+ has its own CPT mirror image, $\bar{\nu}_-$, connected to $\bar{\nu}_+$ by a Lorentz boost:



Thus, there are four states with the same mass. This type of neutrino is referred to as a Dirac neutrino.

If the answer to the above question is *yes*, the neutrino can be described as a particle with two states with the same mass:



In this case, the neutrino is referred to as a Majorana neutrino. In the rest frame of the Majorana neutrino, a CPT transformation of either of the two states simply reverses the spin. A spatial rotation through π radians then returns the particle to itself.

2.2.4 The Quantum Majorana Field

The demonstration of the previous section can now be put into mathematical terms. In the case of Majorana particles, there is no difference between the particles and antiparticles. However, the operators $a(\mathbf{p}, s)$ and $b(\mathbf{p}, s)$ differ by a phase factor. Thus, the quantum field for the Majorana field is given by a plane-wave expansion that satisfies the Dirac equation:

$$\Psi^M(x) = \sum_{\mathbf{p}, s} \sqrt{\frac{m}{E(\mathbf{p})V}} [a(\mathbf{p}, s)u(\mathbf{p}, s)e^{ip \cdot x} + \lambda a^\dagger(\mathbf{p}, s)v(\mathbf{p}, s)e^{-ip \cdot x}] \quad (2.26)$$

where $a(\mathbf{p}, s)$ is the annihilation operator for the particle a , $a^\dagger(\mathbf{p}, s)$ is the creation operator for the particle a , V is a volume associated to the normalisation of the states and λ is the creation phase factor. To see the need to introduce this phase factor, it is necessary to look at the consequences of redefining the phases of one-particle states (which can always be done). The new one particle state is

$$|a'(\mathbf{p}, s)\rangle = e^{i\phi}|a(\mathbf{p}, s)\rangle$$

but

$$|a'(\mathbf{p}, s)\rangle = a'^{\dagger}(\mathbf{p}, s)|0\rangle$$

$$|a(\mathbf{p}, s)\rangle = a^{\dagger}(\mathbf{p}, s)|0\rangle$$

This results in the following relations:

$$a'^{\dagger}(\mathbf{p}, s) = e^{i\phi} a^{\dagger}(\mathbf{p}, s)$$

$$a'(\mathbf{p}, s) = e^{-i\phi} a(\mathbf{p}, s)$$

The conventional Majorana field, Ψ^M , annihilates the one-particle state with a specific matrix element

$$\langle 0|\Psi^M(0)|a(\mathbf{p}, s)\rangle = \sqrt{\frac{m}{E(\mathbf{p})V}} u(\mathbf{p}, s)$$

The new field Ψ'^M that annihilates the new state with the same matrix element is $\Psi'^M = e^{-i\phi}\Psi^M$. With the expansion of Ψ^M the result is

$$\Psi'^M(x) = \sum_{\mathbf{p}, s} \sqrt{\frac{m}{E(\mathbf{p})V}} [a'(\mathbf{p}, s)u(\mathbf{p}, s)e^{ip \cdot x} + \lambda e^{-2i\phi} a'^{\dagger}(\mathbf{p}, s)v(\mathbf{p}, s)e^{-ip \cdot x}] \quad (2.27)$$

Therefore, λ is an arbitrary phase which depends on the set of one-particle states chosen. It will be shown later that this phase factor has important implications for CP violation in the lepton sector.

Other properties of the Majorana field are obtained by looking at C transformations. By definition (see Eq. (A.13) in Appendix A), the charge conjugation transformation for a field is

$$\Psi^C = -i\gamma_2 (\Psi)^* \quad (2.28)$$

Applying this to the Majorana field results in

$$\begin{aligned} -i\gamma_2 (\Psi^M)^* &= -i\gamma_2 \sum_{\mathbf{p}, s} \sqrt{\frac{m}{E(\mathbf{p})V}} [a^{\dagger}(\mathbf{p}, s)u^*(\mathbf{p}, s)e^{-ip \cdot x} \\ &\quad + \lambda^* a(\mathbf{p}, s)v^*(\mathbf{p}, s)e^{ip \cdot x}] \\ &= \lambda^* \Psi^M \end{aligned} \quad (2.29)$$

Therefore, $(\Psi^M)^C = \lambda^* \Psi^M$, which demonstrates that the charge conjugated Majorana field is the original field multiplied by a phase factor and shows that in a free field theory, the Majorana field is an eigenstate of C .

It can also be shown that if a field is its own charge conjugate, then it is a Majorana field. Consider a more general plane-wave expansion:

$$\Psi^D(x) = \sum_{\mathbf{p}, s} \sqrt{\frac{m}{E(\mathbf{p})V}} [a(\mathbf{p}, s)u(\mathbf{p}, s)e^{ip \cdot x} + b^\dagger(\mathbf{p}, s)v(\mathbf{p}, s)e^{-ip \cdot x}] \quad (2.30)$$

With a suitable choice of $b(\mathbf{p}, s)$, this field can describe either a Dirac or a Majorana particle. Suppose that the charge conjugate of the field in Eq. (2.30) is equal to itself, apart from a phase factor. Remembering that the charge conjugation is just the original wave function with a and b switched, it is seen that this would imply that $b(\mathbf{p}, s) = (\text{phase})a(\mathbf{p}, s)$ and Ψ would in fact be a Majorana particle. As a consequence, Ψ is a Majorana field if and only if

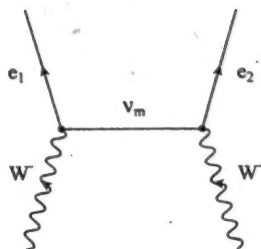
$$\Psi^C = (\text{phase}) \cdot \Psi \quad (2.31)$$

2.2.5 Calculations with Majorana Neutrinos

A spectacular confirmation of the Majorana neutrino mass would be the observation of a neutrinoless double β -decay ($0\nu\beta\beta$). Such a decay is responsible for the nuclear reaction

$$(A, Z) \rightarrow (A, Z + 2) + 2e^-. \quad (2.32)$$

As far as particle physics is concerned, the important contribution is shown in Figure 2.2 where the two W -bosons are initially emitted from a nucleus. Upon inspection of the diagram, it is apparent that the only way in which this process can occur is if the neutrino and the antineutrino are the same particle, that is, if the neutrino is a Majorana particle. However, even if the neutrino is a Majorana particle, the process will be suppressed due to helicity arguments.

Figure 2.2: Basic mechanism for neutrinoless double β -decay

The helicity of a Majorana particle is exactly the same as if the particle was of Dirac type, therefore the neutrino that is emitted at the first vertex would have a positive helicity and the neutrino absorbed at the second vertex would have a negative helicity. As mentioned in Section 2.2.3, the helicity flip can take place by a Lorentz boost if the neutrino has a mass. Thus, the rate at which the process occurs depends on the mass of the neutrino. This mass will actually be the effective mass of the neutrino due to the superposition of mass states for a neutrino of a specific flavour.

The most precise experimental results, obtained with an 11.5 kg sample of ^{76}Ge [22], indicate that the double β -decay lifetime is at least 10^{25} years, which puts an upper limit of 0.4 eV on the effective Majorana mass.¹

In Figure 2.2, ν_m is a mass eigenstate of a Majorana neutrino and thus a summation must be made over all possible ν_m to obtain the total amplitude. The amplitude for a particular eigenstate is proportional to

$$[W^\nu \bar{e} \gamma_\nu (1 - \gamma_5) U_{em} \nu_m] [W^\mu \bar{e} \gamma_\mu (1 - \gamma_5) U_{em} \nu_m] \quad (2.33)$$

To establish the propagator for the Majorana field, the following trick is used:

$$\bar{e} \gamma_\mu (1 - \gamma_5) \nu_m = \overline{(e^c)^c} \gamma_\mu (1 - \gamma_5) (\nu_m^c)^c$$

¹There is some controversy over this result. See [23] for a review.

$$\begin{aligned}
&= (e^c)^T C^{-1} \gamma_\mu (1 - \gamma_5) C (\bar{\nu}_m^c)^T \\
&= -(e^c)^T [C^{-1} \gamma_\mu C - C^{-1} \gamma_\mu \gamma_5 C] (\bar{\nu}_m^c)^T \\
&= -(e^c)^T [\gamma_\mu^T + (\gamma_\mu \gamma_5)^T] (\bar{\nu}_m^c)^T \\
&= -\bar{\nu}_m^c \gamma_\mu (1 + \gamma_5) e^c
\end{aligned}$$

Since ν_m is a Majorana neutrino, the relation $(\Psi^M)^C = \lambda^* \Psi^M$ holds. Thus

$$\nu_m^c = \lambda_m^* \nu_m \quad (2.34)$$

where λ_m is the creation phase factor for the Majorana neutrino field. Therefore,

$$\bar{e} \gamma_\mu (1 - \gamma_5) \nu_m = -\lambda_m \bar{\nu}_m \gamma_\mu (1 + \gamma_5) e^c \quad (2.35)$$

Using this result, the amplitude becomes

$$- [W^\mu \bar{e} \gamma_\mu (1 - \gamma_5) U_{em} \nu_m] [\lambda_m W^\nu \bar{\nu}_m \gamma_\nu (1 + \gamma_5) e^c U_{em}] \quad (2.36)$$

with the contraction of $\nu_m \bar{\nu}_m$ giving the usual fermion propagator, since it satisfies the Dirac equation.

To obtain the wavefunctions for the electrons in the process, recall that the electron has the following structures

$$e \sim \sum [a(\mathbf{p}, s) u(\mathbf{p}, s) + b^\dagger(\mathbf{p}, s) v(\mathbf{p}, s)] \quad (2.37)$$

$$\bar{e} \sim \sum [a^\dagger(\mathbf{p}, s) \bar{u}(\mathbf{p}, s) + b(\mathbf{p}, s) \bar{v}(\mathbf{p}, s)] \quad (2.38)$$

$$e^c \sim \sum [b(\mathbf{p}, s) u(\mathbf{p}, s) + a^\dagger(\mathbf{p}, s) v(\mathbf{p}, s)] \quad (2.39)$$

It is seen that the spinor corresponding to the creation of the e^c is the v . After treating the W -bosons as external particles with polarization vectors ϵ_i^μ , the amplitude becomes

$$\epsilon_1^\mu \bar{u}_1 \gamma_\mu (1 - \gamma_5) \frac{i(\not{q} + M_m)}{q^2 - M_m^2} \gamma_\nu (1 + \gamma_5) v_2 \epsilon_2^\nu \lambda_m U_{em}^2 \quad (2.40)$$

Neglecting the M^2 term compared to the q^2 term in the denominator and summing over all the mass eigenstates, the amplitude becomes

$$A \sim \sum 2\bar{u}_1 \not{\epsilon}_1 \frac{M_m}{q^2} \not{\epsilon}_2 (1 + \gamma_5) v_2 \lambda_m U_{em}^2 \times NF \quad (2.41)$$

where NF represents the nuclear factor involved with the emission of the W -bosons.

Note that the amplitude seems to be dependent on the arbitrary phase factor, λ_m . This will become important in the discussion on CP violation and the number of generations required to generate a complex phase.

2.2.6 CP Violation with Neutrino Mixing

The idea of neutrino mixing was discussed in Section 2.2.2 where the mixing matrix was parametrised by the CKM matrix given in Eq. (2.18). It was mentioned that in the quark sector, three generations of quarks were needed in order to generate a CP-violating phase. The following section will explain that if neutrinos are Majorana particles, only two generations are sufficient for a CP-violating phase to exist.

For N generations of leptons, the lepton mixing matrix, U , can be specified by N^2 independent real parameters due to unitarity. Of these, N are not significant since U changes if we multiply any of the N charged lepton fields by a phase factor. Thus

$$N^2 - N = N(N - 1) \quad (2.42)$$

significant factors remain. Of course the Majorana neutrino fields, ν_m , can also be multiplied by phase factors as discussed in Section 2.2.4. However, when such a multiplication is performed, not only the U_{fm} but also the creation phase factors, λ_m , change, as demonstrated in Section 2.2.4. It was also illustrated in Section 2.2.5 that λ_m may appear in the reaction amplitudes together with

elements of U in ν_m -rephasing-invariant quantities such as

$$\begin{aligned}\omega_{fm} &= \frac{U_{fm}}{U_{fm}^*} \lambda_m \quad f = 1 \dots N \\ \Omega_{fmm'} &= \frac{\omega_{fm}}{\omega_{fm'}} \quad f, m, m' = 1 \dots N\end{aligned}\quad (2.43)$$

which have observable effects. Thus, while the opportunity to shift phase factors out of U by rephasing the Majorana neutrinos is present, only those phase factors that can be removed from U through such rephasing are actually physically significant. Their removal from U does not eliminate them but merely shifts them to the creation phase factor.

The possible number of CP-violating phases can be counted in the lepton sector as in the quark sector. As before, U contains $N(N-1)$ significant parameters, allowing for the fact that the charged lepton fields can be rephased. This number of parameters is not further reduced by the possibility of rephasing the neutrino fields because such rephasing only moves a significant phase factor from U to λ_m . If CP is conserved, the mixing matrix can be chosen to be real. Due to the orthogonality of such a mixing matrix, there are N^2 real numbers subject to

$$N + \frac{1}{2}N(N-1) \quad (2.44)$$

constraints. U then contains

$$\frac{1}{2}N(N-1) \quad (2.45)$$

parameters. Thus, the number of CP-violating phases (CPVP) in the lepton sector is the number of parameters in the general case minus the number of parameters when CP is conserved:

$$\text{CPVP} = \frac{1}{2}N(N-1) \quad (2.46)$$

Thus, for two generations of leptons, $N = 2$ and CPVP = 1 and it is seen that there is a CP-violating phase present with only two generations. A way to

parameterize the CP-violating phase is with the matrix [24]

$$U = \begin{pmatrix} c & se^{i\delta} \\ -se^{-i\delta} & c \end{pmatrix} \quad (2.47)$$

where c and s are the cosine and sine of the mixing angle θ and δ is the CP-violating phase.

2.2.7 Majorana Mass Generation

It is instructive to look at how the most general mass terms can be constructed for fermion fields in a Lagrangian. Separating a fermion wavefunction into left- and right-handed pieces via $\psi = \psi_L + \psi_R$, the usual Dirac mass term can be constructed:

$$\begin{aligned} \bar{\psi}\psi &= (\bar{\psi}_L + \bar{\psi}_R)(\psi_L + \psi_R) \\ &= \psi^\dagger a_L \gamma^0 a_L \psi + \psi^\dagger a_L \gamma^0 a_R \psi + \psi^\dagger a_R \gamma^0 a_L \psi + \psi^\dagger a_R \gamma^0 a_R \psi \\ &= \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \end{aligned} \quad (2.48)$$

Here, a_L and a_R project onto the left- and right-handed pieces of ψ such that $a_{L/R} \gamma^0 = \gamma^0 a_{R/L}$ and $a_L a_R = a_R a_L = 0$ have been used. Thus, for a Dirac type particle, the mass term for the Lagrangian is

$$\begin{aligned} \mathcal{L}_D &= D \bar{\psi}\psi \\ &= D (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \end{aligned} \quad (2.49)$$

where D is the Dirac mass. It can be seen that the Lagrangian mass term connects the left- and right-handed pieces of the same field.

For the Majorana field, it has been demonstrated in Section 2.2.3 that the right-handed field can be written as the charge conjugate field of the left-handed field and vice versa. Thus, there exist fields which satisfy

$$\chi = \psi_L + \psi_L^c \quad (2.50)$$

$$\omega = \psi_R + \psi_R^c \quad (2.51)$$

and are self-conjugate fields. From these relations, mass terms can be obtained of the form:

$$\begin{aligned} \bar{\chi}\chi &= (\bar{\psi}_L + \bar{\psi}_L^c)(\psi_L + \psi_L^c) \\ &= \bar{\psi}_L\psi_L + \bar{\psi}_L\psi_L^c + \bar{\psi}_L^c\psi_L + \bar{\psi}_L^c\psi_L^c \\ &= \bar{\psi}_L\psi_L^c + \bar{\psi}_L^c\psi_L \end{aligned} \quad (2.52)$$

Similarly,

$$\bar{\omega}\omega = \bar{\psi}_R\psi_R^c + \bar{\psi}_R^c\psi_R \quad (2.53)$$

Thus, the Majorana mass terms that appear in the Lagrangian will be of the form

$$\mathcal{L}_\chi = A(\bar{\psi}_L\psi_L^c + \bar{\psi}_L^c\psi_L) \quad (2.54)$$

$$\mathcal{L}_\omega = B(\bar{\psi}_R\psi_R^c + \bar{\psi}_R^c\psi_R) \quad (2.55)$$

and it is seen that the mass term for a Majorana type particle connects charge conjugate fields.

When both the Dirac and Majorana mass terms are present, we have

$$\mathcal{L}_{DM} = D\bar{\psi}\psi + A\bar{\chi}\chi + B\bar{\omega}\omega \quad (2.56)$$

By noting that

$$\begin{aligned} \frac{1}{2}[\bar{\chi}\omega + \bar{\omega}\chi] &= \frac{1}{2}\left[(\bar{\psi}_L + \bar{\psi}_L^c)(\psi_R + \psi_R^c) + (\bar{\psi}_R + \bar{\psi}_R^c)(\psi_L + \psi_L^c)\right] \\ &= \frac{1}{2}[\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R^c + \bar{\psi}_R\psi_L^c] \\ &= \bar{\psi}\psi \end{aligned} \quad (2.57)$$

we can write the mass terms for the Lagrangian as

$$\mathcal{L}_{MD} = \frac{1}{2}D(\bar{\chi}\omega + \bar{\omega}\chi) + A\bar{\chi}\chi + B\bar{\omega}\omega \quad (2.58)$$

$$= \begin{pmatrix} \bar{\chi} & \bar{\omega} \end{pmatrix} \begin{pmatrix} A & \frac{1}{2}D \\ \frac{1}{2}D & B \end{pmatrix} \begin{pmatrix} \chi \\ \omega \end{pmatrix} \quad (2.59)$$

Thus, the most general mass term for a four-component fermion field actually describes two Majorana particles with distinct masses. The usual four-component Dirac field formalism can be recovered in the limit that $A = B = 0$. Looking at the mass terms for the Majorana fields, it is seen that they violate any additive quantum number that the field carries. Since all elementary fermions, except for neutrinos, have charge, they must have $A = B = 0$. For neutrinos, Majorana mass terms violate lepton number by two units. This violation leads to processes that do not occur in the Standard Model and is a candidate to induce an electric dipole moment for the electron.

2.2.8 Seesaw Mechanism

The neutrino mass matrix will have two eigenvectors, ν' and N , which represent two particles with masses $M_{\nu'}$ and M_N . Recall that a Dirac field from which the original Lagrangian was written, Eq. (2.49), consisted of four states corresponding to two spin states for the particle and two spin states for the antiparticle. Since the new result has two particles of different masses, each of the particles must have only two spin states so it is expected that each of the particles is a Majorana particle.

It is now important to see how the physical masses are constructed. In order to accomplish this, consider a left-right symmetric model which is described by an $SU(2)_L \times SU(2)_R \times U(1)$ gauge group. The gauge group is first broken down to $SU(2)_L \times U(1)$ and the W_R field acquires a mass M_{W_R} which is of the order of the symmetry breaking scale, $\langle \Phi \rangle_R$. (Recall that during spontaneous symmetry breaking, the scalar Higgs field acquires a vacuum expectation value, $\langle \Phi \rangle_0$, providing the mass term, $\langle \Phi \rangle_0 \bar{\psi} \psi$.)

The remaining gauge group is broken down to $U(1)$ and the W_L field acquires a mass, M_{W_L} . This is the well known W -boson whose mass is measured to be

~ 80 GeV [25]. Since there has been no experimental evidence for a right-handed W -boson, it can be assumed that the symmetry breaking scale of $SU(2)_R$, and hence M_{W_R} , will be much larger than that of $SU(2)_L$.

With this in mind, Yukawa couplings can now be constructed. Since the model being used is a left-right symmetric model, the neutrino fields, ψ_L and ψ_R , will have $SU(2)_L \times SU(2)_R \times U(1)$ quantum numbers. Thus, the ψ_L belongs to an $SU(2)_L$ doublet

$$\begin{pmatrix} \psi_L \\ e_L \end{pmatrix} \quad (2.60)$$

with a weak isospin assignment of $I_L = 1/2$ and $I_R = 0$. However, the ψ_R now also belongs to an $SU(2)_R$ doublet

$$\begin{pmatrix} \psi_R \\ e_R \end{pmatrix} \quad (2.61)$$

with weak isospin assignment of $I_R = 1/2$ and $I_L = 0$.

Bilinear products can now be constructed involving ψ_L and ψ_R . The usual terms given in Eqs. (2.49), (2.54) and (2.55) are forbidden due to the fact that these terms are not $SU(2)_L \times SU(2)_R \times U(1)$ scalars. For example, $\bar{\psi}_R \psi_L$ has $I_L = 1/2$ and $I_R = 1/2$, $(\bar{\psi}_L)^c \psi_L$ has $I_L = 1$ and $I_R = 0$ and $(\bar{\psi}_R)^c \psi_R$ has $I_L = 0$ and $I_R = 1$. In order to produce terms with $I_L = I_R = 0$, the introduction of Higgs fields is necessary. The introduction of Φ with $I_L = 1/2$ and $I_R = 1/2$, Δ_L with $I_L = 1$ and $I_R = 0$ and Δ_R with $I_L = 0$ and $I_R = 1$ allows for terms such as

$$\bar{\psi}_R \Phi \psi_L, \quad (\bar{\psi}_L)^c \Delta_L \psi_L, \quad (\bar{\psi}_R)^c \Delta_R \psi_R \quad (2.62)$$

to be built with the proper isospin invariance. If the fields ψ, χ, ω are now taken to be the neutrino fields and the Higgs fields acquire vacuum expectation values, there will be Dirac and Majorana neutrino mass terms with

$$D \sim \langle \Phi \rangle \quad (2.63)$$

$$A \sim \langle \Delta_L \rangle \quad (2.64)$$

$$B \sim \langle \Delta_R \rangle \quad (2.65)$$

The size of the mass scales can now be physically motivated. First, the expectation value of Δ_L will affect the well known ratio $\rho = M_W^2 / \cos^2 \theta_W M_Z^2$, where θ_W is the weak mixing angle. Thus, Δ_L must vanish.

Next, the same spontaneous symmetry breaking which leads to the neutrino Dirac masses should also lead to the Dirac masses of the charged leptons. As stated earlier, the charged fermions cannot have a Majorana mass term. Therefore, the Dirac masses for the charged leptons are their physical masses and the neutrino Dirac mass, D , is expected to be of the same order of the "observed" masses of the related charged leptons and quarks.

Finally, since $\langle \Delta_R \rangle$ gives the W_R its large mass, it is expected to be very large such that $B \gg D$.

The neutrino mass matrix now takes on the form

$$[M] = \begin{pmatrix} 0 & M_D \\ M_D & M_R \end{pmatrix} \quad (2.66)$$

where M_D is the Dirac mass, M_R is the $SU(2)_R$ symmetry breaking scale such that $M_R \gg M_D$.

The eigenvalues of this matrix are

$$M_N \sim M_R, \quad M_{\nu'} \sim -\frac{M_D^2}{M_R} \quad (2.67)$$

In terms of the fields χ and ω , the eigenvector fields can be written as

$$N \simeq \omega + \frac{M_D}{M_R} \chi \quad (2.68)$$

$$\nu' \simeq \chi - \frac{M_D}{M_R} \omega \quad (2.69)$$

Since the neutrino field has a negative mass, as seen in Eq. (2.67), the physical field is given by

$$\nu = \gamma^5 \nu' \quad (2.70)$$

so that $\bar{\nu}'\nu' = -\bar{\nu}\nu$. If M_D is a typical quark or charged lepton mass, as expected, the masses obey the seesaw relation:

$$M_\nu M_N = M_D^2 \quad (2.71)$$

To recap this section, it has been shown that with the presence of Majorana mass terms a neutrino mass matrix can be formed. The eigenvectors of this matrix correspond to a light neutral lepton, ν , as well as a heavy neutral lepton, N , with masses M_D^2/M_R and M_R , respectively.

To see how these particles interact, note that the assignment of ψ_L and e_L to a left-handed weak isospin doublet means that the W_L -boson is coupled to

$$\bar{e}_L \gamma^\mu \psi_L = \bar{e}_L \gamma^\mu \chi_L \quad (2.72)$$

By expressing ω in terms of the ν' and N fields given by Eq. (2.69), the interaction becomes

$$\bar{e}_L \gamma^\mu \chi_L = \bar{e}_L \gamma^\mu \left(\nu'_L + \frac{M_D}{M_N} N_L \right) \quad (2.73)$$

where the fact that $M_D \ll M_N$ has been used to simplify the coupling. Noting that $\nu'_L = \nu_L$, the left-handed weak current can then be written as

$$\bar{e}_L \gamma^\mu \psi_L = \bar{e}_L \gamma^\mu \left(\nu_L + \frac{M_D}{M_N} N_L \right) \quad (2.74)$$

It is seen that the neutrinos ψ_L and ψ_R of one generation, which couple to W_L and W_R respectively, are a linear combination of the two mass eigenstates ν and N .

2.3 The Electric Dipole Moment

2.3.1 Non-Relativistic Limit of the Electric Dipole Moment

The structure of the electric dipole moment operator can be recognized by evaluating the non-relativistic limit and comparing the result to the well-known structure given in quantum mechanics. The non-relativistic limit will also serve to show why CP violation occurs in an electric dipole moment.

The process begins by noting that the magnetic dipole term is given by $\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi$. The non-relativistic limit can be illustrated by calculating this quantity explicitly. First, the above term can be rewritten as

$$\begin{aligned}\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi &= \frac{i}{2}\bar{\psi}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]F_{\mu\nu}\psi \\ &= i\bar{\psi}\gamma^\mu\gamma^\nu F_{\mu\nu}\psi\end{aligned}\quad (2.75)$$

where the fact that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $F^{\mu\nu}g_{\mu\nu} = 0$ have been used.

Looking at $\gamma^\mu\gamma^\nu F_{\mu\nu}$ and choosing a specific representation for the field tensor and the γ matrices (see Appendix A), the result is

$$\begin{aligned}\gamma^\mu\gamma^\nu F_{\mu\nu} &= (\gamma^0\gamma^1 - \gamma^1\gamma^0)F_{01} + (\gamma^0\gamma^2 - \gamma^2\gamma^0)F_{02} \\ &+ (\gamma^0\gamma^3 - \gamma^3\gamma^0)F_{03} + (\gamma^1\gamma^2 - \gamma^2\gamma^1)F_{12} \\ &+ (\gamma^1\gamma^3 - \gamma^3\gamma^1)F_{13} + (\gamma^2\gamma^3 - \gamma^3\gamma^2)F_{23}\end{aligned}\quad (2.76)$$

since $F_{\mu\nu}$ is an antisymmetric tensor.

Working out the matrix multiplication, the result is

$$\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi = 2\bar{\psi}\begin{pmatrix} \mathbf{E} \cdot \boldsymbol{\sigma} - i\mathbf{B} \cdot \boldsymbol{\sigma} & 0 \\ 0 & -\mathbf{E} \cdot \boldsymbol{\sigma} - i\mathbf{B} \cdot \boldsymbol{\sigma} \end{pmatrix}\psi\quad (2.77)$$

To obtain the non-relativistic limit, look at the wavefunction in the rest frame of the particle, which has a Dirac spinor of the form

$$u(p_0) = \begin{pmatrix} \xi \\ \xi \end{pmatrix} \quad (2.78)$$

for any numerical two-component spinor ξ . This leads to

$$\begin{aligned} \bar{\psi}(x) \sigma^{\mu\nu} F_{\mu\nu} \psi(x) &= 2 \begin{pmatrix} \xi^\dagger & \xi^\dagger \end{pmatrix} \begin{pmatrix} \mathbf{E} \cdot \boldsymbol{\sigma} - i\mathbf{B} \cdot \boldsymbol{\sigma} & 0 \\ 0 & -\mathbf{E} \cdot \boldsymbol{\sigma} - i\mathbf{B} \cdot \boldsymbol{\sigma} \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \\ &= -4i\xi^\dagger \mathbf{B} \cdot \boldsymbol{\sigma} \xi \end{aligned} \quad (2.79)$$

which is proportional to the magnetic dipole moment of a non-relativistic particle.

Notice now that if γ_5 is multiplied into Eq. (2.77), the result is

$$\begin{aligned} \bar{\psi} \gamma_5 \sigma^{\mu\nu} F_{\mu\nu} \psi &= 2\bar{\psi} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{E} \cdot \boldsymbol{\sigma} - i\mathbf{B} \cdot \boldsymbol{\sigma} & 0 \\ 0 & -\mathbf{E} \cdot \boldsymbol{\sigma} - i\mathbf{B} \cdot \boldsymbol{\sigma} \end{pmatrix} \psi \\ &= 2\bar{\psi} \begin{pmatrix} i\mathbf{B} \cdot \boldsymbol{\sigma} - \mathbf{E} \cdot \boldsymbol{\sigma} & 0 \\ 0 & -i\mathbf{B} \cdot \boldsymbol{\sigma} - \mathbf{E} \cdot \boldsymbol{\sigma} \end{pmatrix} \psi \\ &\xrightarrow{\text{NRlimit}} -4\xi^\dagger \mathbf{E} \cdot \boldsymbol{\sigma} \xi \end{aligned} \quad (2.80)$$

In this case, the non-relativistic limit is proportional to the electric dipole moment for the field. Thus, the term that will contribute to the electric dipole moment is

$$\text{EDM} = \bar{\psi} \gamma_5 \sigma^{\mu\nu} F_{\mu\nu} \psi \quad (2.81)$$

The non-relativistic limit also provides an insight into how an electric dipole moment is a CP-violating effect while the magnetic dipole moment is not.

The non-relativistic limits are given in Eqs. (2.79) and (2.80), with

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned} \quad (2.82)$$

It has been shown already in Section 2.1 that angular momentum does not change sign under a parity transformation and since there is no charge dependence in the spin

$$\sigma \xrightarrow{\text{CP}} \sigma \quad (2.83)$$

However, the fields transform as

$$\begin{aligned} \mathbf{E} &\xrightarrow{\text{CP}} -[(-\nabla)(\phi)] - \frac{\partial(-\mathbf{A})}{\partial t} = -\mathbf{E} \\ \mathbf{B} &\xrightarrow{\text{CP}} -\nabla \times (-\mathbf{A}) = \mathbf{B} \end{aligned} \quad (2.84)$$

since the scalar potential is invariant under CP and the vector potential changes sign. The overall effect of CP on the nonrelativistic limits is given by

$$\begin{aligned} \sigma \cdot \mathbf{E} &\xrightarrow{\text{CP}} -\sigma \cdot \mathbf{E} \\ \sigma \cdot \mathbf{B} &\xrightarrow{\text{CP}} \sigma \cdot \mathbf{B} \end{aligned} \quad (2.85)$$

which indicates that the magnetic dipole moment is invariant under CP transformations, but the electric dipole moment is not.

2.3.2 EDM Projector

To calculate the electric dipole moment of the electron, a projection operator can be derived to project out the electric dipole term from the amplitude. To find the projector, consider the diagram in Figure 2.3.

The external momenta are defined in the diagram. As a further simplification, denote $t = q^2$ and $p = \frac{1}{2}(p_1 + p_2)$. If the external fermions are "on-shell", $p_1^2 = p_2^2 = m^2$, where m is the mass of the fermion. With the definitions above, it is easy to show the following relationships:

$$p_1 \cdot p_2 = m^2 - \frac{t}{2} \quad (2.86)$$

$$p \cdot p_1 = p \cdot p_2 = p^2 = m^2 - \frac{t}{4} \quad (2.87)$$

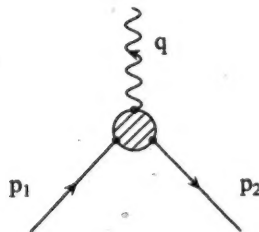


Figure 2.3: Vertex diagram for the EDM of the electron

$$q \cdot p = 0 \quad (2.88)$$

$$q \cdot p_1 = -q \cdot p_2 = \frac{t}{2} \quad (2.89)$$

$$p_1 = p + \frac{q}{2} \quad (2.90)$$

$$p_2 = p - \frac{q}{2} \quad (2.91)$$

Define an operator

$$N_\mu = (\not{p}_1 + m)\gamma_5 \left(g_4 \gamma_\mu - \frac{1}{m} g_5 \not{p}_\mu - \frac{1}{m} g_6 \not{q}_\mu \right) (\not{p}_2 + m) \quad (2.92)$$

which, for appropriate choices of g_4 , g_5 and g_6 will project out the EDM portion of the amplitude.

Also, consider the general form of a matrix element of a current between spin-1/2 fermions that can be generated [26]

$$\begin{aligned} \langle \alpha_f | M^\mu | \alpha_i \rangle &= \bar{u}_f(p_2) \left[F_1(t) \gamma^\mu - \frac{i}{2m} F_2(t) \sigma^{\mu\nu} q_\nu + \frac{1}{m} F_3(t) q^\mu \right. \\ &\quad \left. + \gamma_5 \left(G_1(t) \gamma^\mu - \frac{i}{2m} G_2(t) \sigma^{\mu\nu} q_\nu + \frac{1}{m} G_3(t) q^\mu \right) \right] u_i(p_1) \end{aligned} \quad (2.93)$$

where

$$\sigma^{\mu\nu} = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \quad (2.94)$$

To find the projector for the electric dipole moment, calculate $\text{Tr}[N_\mu M^\mu]$. Immediately, the terms with F_1 , F_2 and F_3 are decoupled. Next, select g_4 , g_5 and g_6 so as to isolate the $G_2(t)$ term.

The result of the trace is

$$\begin{aligned} \text{Tr}[N_\mu M^\mu] &= 4 \left[\left((2-D) \left(m^2 - \frac{t}{2} \right) - Dm^2 \right) g_4 - tg_6 \right] G_1(t) \\ &+ \left[\frac{2}{m^2} g_5 t \left(m^2 - \frac{t}{4} \right) \right] G_2(t) \\ &- \left[4tg_4 + \frac{2t^2}{m^2} g_6 \right] G_3(t) \end{aligned} \quad (2.95)$$

where it will be convenient to allow the number of space-time dimensions, D , to remain unspecified.

In order to isolate the electric dipole term $G_2(t)$, the following set of equations must be solved:

$$\left[(2-D) \left(m^2 - \frac{t}{2} \right) - Dm^2 \right] g_4 - tg_6 = 0 \quad (2.96)$$

$$\frac{2}{m^2} g_5 t \left(m^2 - \frac{t}{4} \right) = 1 \quad (2.97)$$

$$4tg_4 + \frac{2t^2}{m^2} g_6 = 0 \quad (2.98)$$

It can be seen that the g_4 and g_6 terms decouple from the g_5 term and thus this set of equations can be satisfied easily if the constants are chosen as

$$g_4 = g_6 = 0 \quad (2.99)$$

$$g_5 = \frac{m^2}{2t \left(m^2 - \frac{t}{4} \right)} \quad (2.100)$$

Therefore, to isolate the electric dipole moment term of a process, use the following projector

$$G_2(t) = \frac{-m}{2t \left(m^2 - \frac{t}{4} \right)} \text{Tr}[(\not{p}_1 + m)\gamma_5(\not{p}_2 + m)\not{p}_\mu M^\mu] \quad (2.101)$$

on any amplitude M^μ .

There is a slight problem, though. The actual electric dipole moment is found in the limit as $t \rightarrow 0$, a limit in which Eq. (2.101) appears to diverge. To handle

the $t \rightarrow 0$ limit correctly, the amplitude is expanded about the point $q = 0$.

$$\begin{aligned} M^\mu(p, q) &= M^\mu(p, 0) + q_\nu \frac{\partial M^\mu}{\partial q_\nu} \Big|_{q=0} \\ &= V^\mu + q_\nu T^{\mu\nu} \end{aligned} \quad (2.102)$$

These terms will now be evaluated separately. Starting with the V^μ term,

$$T_V = \frac{-m}{2t(m^2 - \frac{t}{4})} \text{Tr}[(\not{p}_1 + m)\gamma_5(\not{p}_2 + m)p_\mu V^\mu] \quad (2.103)$$

At this stage, it is necessary to rewrite this trace in terms of the momenta p and q .

$$T_V = \frac{-m}{2t(m^2 - \frac{t}{4})} \text{Tr} \left[\left([\not{p} + m] + \frac{\not{q}}{2} \right) \gamma_5 \left([\not{p} + m] - \frac{\not{q}}{2} \right) p_\mu V^\mu \right] \quad (2.104)$$

Notice here that q averaged over all spatial directions is zero but requiring $p^\mu \langle q_\mu q_\nu \rangle = 0$ and $g^{\mu\nu} \langle q_\mu q_\nu \rangle = q^2$ leads to the relation

$$\langle q_\mu q_\nu \rangle = \frac{q^2}{D-1} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \quad (2.105)$$

With these results, all the terms in T_V that are linear in q vanish and what is left are the terms that are independent of q and terms that are quadratic in q .

Looking at the terms that are independent of q :

$$\begin{aligned} T_{V_0} &= \frac{-m}{2t(m^2 - \frac{t}{4})} \text{Tr}[(\not{p} + m)\gamma_5(\not{p} + m)p_\mu V^\mu] \\ &= \frac{-m}{8(m^2 - \frac{t}{4})} \text{Tr}[\gamma_5 p_\mu V^\mu] \end{aligned} \quad (2.106)$$

Now looking at the terms quadratic in q :

$$\begin{aligned} T_{V_{q^2}} &= \frac{m}{8t(m^2 - \frac{t}{4})} \text{Tr}[\not{q}\gamma_5\not{q}p_\mu V^\mu] \\ &= \frac{-m}{8(m^2 - \frac{t}{4})} \text{Tr}[\gamma_5 p_\mu V^\mu] \end{aligned} \quad (2.107)$$

The result, then, for the first term in the amplitude expansion is

$$T_V = \frac{-m}{4(m^2 - \frac{t}{4})} \text{Tr}[\gamma_5 p_\mu V^\mu] \quad (2.108)$$

Now the focus is on the the second term in the q -expansion

$$T_T = \frac{-m}{2t(m^2 - \frac{t}{4})} \text{Tr} \left[\left([\not{p} + m] + \frac{\not{q}}{2} \right) \gamma_5 \left([\not{p} + m] - \frac{\not{q}}{2} \right) p_\mu q_\nu T^{\mu\nu} \right] \quad (2.109)$$

In this case, there are no terms independent of q , but again, the terms linear and cubic in q will vanish. Thus, only the q^2 terms will remain, resulting in the following:

$$\begin{aligned} T_T &= \frac{-m}{2t(m^2 - \frac{t}{4})} \text{Tr} \left[\left(\frac{\not{q}}{2} \gamma_5 [\not{p} + m] - [\not{p} + m] \gamma_5 \frac{\not{q}}{2} \right) p_\mu q_\nu T^{\mu\nu} \right] \\ &= \frac{-m}{2(D-1)(m^2 - \frac{t}{4})} \text{Tr} \left[\left(\frac{p_\nu \not{p}}{p^2} - \gamma_\nu \right) (\not{p} + m) p_\mu T^{\mu\nu} \gamma_5 \right] \end{aligned} \quad (2.110)$$

The final result for the projector of the electric dipole moment is given by

$$G_2(0) = \frac{-p_\mu}{4m} \text{Tr} \left[\gamma_5 V^\mu + \frac{2}{D-1} \left(\frac{p_\nu \not{p}}{p^2} - \gamma_\nu \right) (\not{p} + m) T^{\mu\nu} \gamma_5 \right] \quad (2.111)$$

which is explicitly finite in the limit $t \rightarrow 0$.

Chapter 3

Details of the Calculation

3.1 Diagrams and Amplitudes

In the unitary gauge, the electric dipole moment of the electron will have contributions at two loops from the diagrams shown in Figure 3.1, where the photon can couple to either of the W 's or the internal charged lepton, l .

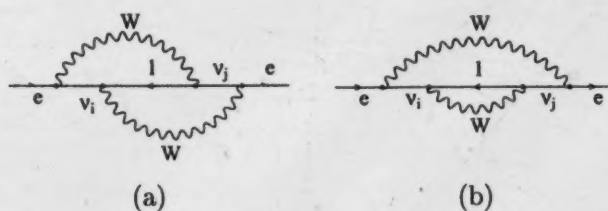


Figure 3.1: Two loop contributions to the electron EDM

It can be seen that in order for these processes to proceed, it is necessary for the neutrinos to be Majorana particles. In fact, if the internal lepton is an electron, the two-loop diagrams are just two neutrinoless double β -decay diagrams connected together. Diagrams similar to Figure 3.1 (b) which have the charge of the internal lepton running in the clockwise direction can also proceed if the neutrino is a Dirac particle, however, these diagrams provide no contribution

[9]. As shown in Section 2.3.2, in order to calculate the EDM, it is necessary to find the amplitude of these diagrams in the limit that the momentum of the photon goes to zero. In order to be able to express the amplitude of the Feynman diagrams for the EDM, it is useful to generate the Feynman rules based on the method used in Section 2.2.5. To illustrate the method, a particular diagram for the electric dipole moment of the electron will be used, as given in Figure 3.2.

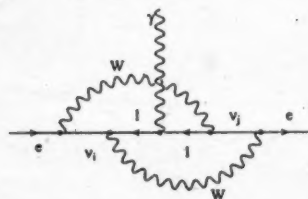


Figure 3.2: Contributing diagram to the electron EDM

The diagram is redrawn as in Figure 3.3 so that the original diagram is obtained by connecting the charged leptons l .

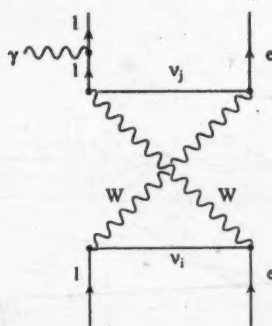


Figure 3.3: Contributing diagram to the electron EDM

The procedure now is to look at the individual interactions to form the amplitude:

$$A \propto [\epsilon^\mu \bar{u}_l \gamma_\mu u_l] [W^\beta \bar{u}_l \gamma_\beta (1 - \gamma_5) \psi_j U_{lj}^*] [W^\delta \bar{u}_e \gamma_\delta (1 - \gamma_5) \psi_j U_{ej}^*]$$

$$\times [W^\alpha \bar{\psi}_i \gamma_\alpha (1 - \gamma_5) u_l U_{li}] [W^\gamma \bar{\psi}_i \gamma_\gamma (1 - \gamma_5) u_e U_{ei}] \quad (3.1)$$

where $u_{e,i}$ and $\psi_{i,j}$ are the spinors for the charged leptons and the neutrinos, respectively. The next step is to rewrite the second term using the properties of the charge conjugation matrix listed in Appendix A:

$$\begin{aligned} \bar{u}_l \gamma_\beta (1 - \gamma_5) \psi_j &= \overline{(u_l^c)^c} \gamma_\beta (1 - \gamma_5) (\psi_j^c)^c \\ &= -(u_l^c)^T C^{-1} \gamma_\beta (1 - \gamma_5) C (\bar{\psi}_j^c)^T \\ &= (u_l^c)^T (\gamma_\beta^T + \gamma_5^T \gamma_\beta^T) (\bar{\psi}_j^c)^T \\ &= -\bar{\psi}_j \gamma_\beta (1 + \gamma_5) v_l \end{aligned} \quad (3.2)$$

where the fact that the charge conjugate of a Majorana neutrino is itself and that the fermion operators anticommute have been used. Similarly, the first and fourth terms can be rewritten

$$\bar{u}_l \gamma_\mu u_l = -\bar{v}_l \gamma_\mu v_l \quad (3.3)$$

$$\bar{\psi}_i \gamma_\alpha (1 - \gamma_5) u_l = -\bar{v}_l \gamma_\alpha (1 + \gamma_5) \psi_i \quad (3.4)$$

As seen in Section 2.2.8, the Majorana field is a mixture of two mass eigenstates, ν and N . Thus, four amplitudes can be obtained. In this project, only the cases where both Majorana particles are heavy or both are light will be calculated. Because every term in square brackets in Eq. (3.1) is just a number, the amplitude can be written as

$$\begin{aligned} A_1 &\propto -\epsilon^\mu \bar{u}_e \gamma_\delta (1 - \gamma_5) [\nu_j \bar{\nu}_j] \gamma_\beta (1 + \gamma_5) [v_l \bar{v}_l] \gamma_\mu [v_l \bar{v}_l] \gamma_\alpha (1 + \gamma_5) \\ &\quad \times [\nu_i \bar{\nu}_i] \gamma_\gamma (1 - \gamma_5) u_e W^\delta W^\beta W^\alpha W^\gamma \end{aligned} \quad (3.5)$$

$$\begin{aligned} A_2 &\propto -\left(\frac{M_{D_i}}{M_{N_i}}\right)^2 \left(\frac{M_{D_j}}{M_{N_j}}\right)^2 \epsilon^\mu \bar{u}_e \gamma_\delta (1 - \gamma_5) [N_j \bar{N}_j] \gamma_\beta (1 + \gamma_5) [v_l \bar{v}_l] \gamma_\mu [v_l \bar{v}_l] \\ &\quad \times \gamma_\alpha (1 + \gamma_5) [N_i \bar{N}_i] \gamma_\gamma (1 - \gamma_5) u_e W^\delta W^\beta W^\alpha W^\gamma \end{aligned} \quad (3.6)$$

The result of the recombination is that a Wick contraction can now take place between u_l and $\bar{u}_l$, ν and $\bar{\nu}$, and N and $\bar{N}$ to give rise to the propagators. Also,

the propagators for the W -bosons are present. Using the momentum assignments in Figure 3.4, the amplitude becomes

$$\begin{aligned}
 A = & -Z\epsilon^\mu(q) \int [d^D k_1] [d^D k_2] \bar{u}_e(p') \gamma_\delta (1 - \gamma_5) \frac{(\not{p}_4 + m_j)}{(p_4^2 - m_j^2)} \gamma_\beta (1 + \gamma_5) \\
 & \times \frac{(\not{p}_3 - m_l)}{(p_3^2 - m_l^2)} \gamma_\mu \frac{(\not{p}_2 - m_l)}{(p_2^2 - m_l^2)} \gamma_\alpha (1 + \gamma_5) \frac{(\not{p}_1 + m_i)}{(p_1^2 - m_i^2)} \gamma_\gamma (1 - \gamma_5) u_e(p) \\
 & \times \frac{\left(g^{\alpha\delta} - \frac{k_1^\alpha k_1^\delta}{M_W^2}\right) \left(g^{\beta\gamma} - \frac{k_2^\beta k_2^\gamma}{M_W^2}\right)}{(k_1^2 - M_W^2) (k_2^2 - M_W^2)} \quad (3.7)
 \end{aligned}$$

where the factor of Z accounts for the Majorana mixing factor and the masses $m_{i,j}$ are unspecified. It can be seen that the interaction between the photon and the internal charged fermion introduces an extra minus sign that is not present in diagrams in which the photon couples to the W . Using the same technique,

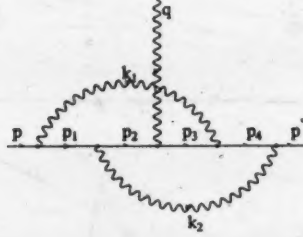


Figure 3.4: Momentum assignment for the EDM calculation

amplitudes can be written for the diagrams with the photon coupling to the W -bosons.

From Figure 3.4, momentum conservation implies that $p_1 = p + k_1$, $p_2 = p + k_1 + k_2$, $p_3 = p + k_1 + k_2 - q$ and $p_4 = p + k_2 - q$.

3.2 Computational Methods

In order to evaluate the diagrams, the method of asymptotic expansions is used. For a review on this subject see [27]. This method basically amounts to expanding

the Feynman diagrams in small masses and momenta. There are five steps to this procedure, as outlined in [28]. The first step is to identify all the large and small external scales of the integrals. Second, divide the integration volume into regions so that the momentum flow through any of the internal lines is of the order of one of the external scales. More specifically, the statement $k \sim M$ asserts that $k \ll m^2$ in Euclidean space. Third, perform Taylor expansions within every region for any individual denominator factors (propagators) where the terms within these factors depend differently on the external scales. Fourth, integrate the expanded integrals from every region over the initial integration volume—in other words, ignoring the constraints that identify the region. This can be done since scaleless integrals are zero in dimensional regularization (For more on dimensional regularisation, see Appendix C.3). The final stage is to add up all contributions from each region to obtain the final result. This method was used to obtain the EDM of the electron for three different limiting cases of the neutrino mass. These calculations are facilitated by the use of the symbolic algebra programming language FORM [29]. A detailed illustration of the calculational methods will now be given.

3.2.1 EDM with Small Neutrino Masses

In this section, it will be assumed that the neutrino masses are quite small, that is, $m_{i,j} \ll m_e, p, m_\mu, M_W$.

The first region that will be considered is the region where $k_1 \sim k_2 \sim M_W$. Since the large scales are identified, the small parameters are p, m_e, m_l, m_i, m_j . A momentum assignment can be chosen such that the integrals left are in the form of

$$\int \frac{[d^D k_1] [d^D k_2] f(k_1^2, k_2^2, k_1 \cdot k_2, k_1 \cdot p, k_2 \cdot p)}{((k_1 - p)^2 + M_W^2)^a (k_1^2 + m_i^2)^b ((k_1 + k_2 - p)^2 + m_l^2)^c}$$

$$\times \frac{1}{(k_2^2 + m_j^2)^d ((k_2 - p)^2 + M_W^2)^e} \quad (3.8)$$

where a, b, c, d, e are integers. Since this region has $k_1 \sim k_2 \sim M_W$, the propagators can be expanded as follows:

$$\begin{aligned} \frac{1}{(k_{1,2} - p)^2 + M_W^2} &= \frac{1}{k_{1,2}^2 + M_W^2} \sum_{n=0}^{\infty} \left(\frac{2k_{1,2} \cdot p - p^2}{k_{1,2}^2 + M_W^2} \right)^n \\ \frac{1}{k_{1,2}^2 + m_{ij}^2} &= \frac{1}{k_{1,2}^2} \sum_{n=0}^{\infty} \left(\frac{-m_{ij}^2}{k_{1,2}^2} \right)^n \\ \frac{1}{(k_1 + k_2 - p)^2 + m_l^2} &= \frac{1}{(k_1 + k_2)^2} \sum_{n=0}^{\infty} \left(\frac{2(k_1 + k_2) \cdot p - p^2 - m_l^2}{(k_1 + k_2)^2} \right)^n \end{aligned} \quad (3.9)$$

To simplify the numerator, an average over the directions of p is performed, since the result is a scalar and should depend only on $p^{2n} = -m_e^{2n}$. All that is remaining are integrals of the form

$$\int \frac{[d^D k_1] [d^D k_2] f'(k_1^2, k_2^2, k_1 \cdot k_2)}{k_1^{2a} (k_1^2 + M_W^2)^b (k_1 + k_2)^{2c} (k_2^2 + M_W^2)^d k_2^{2e}} \quad (3.10)$$

This integral can be evaluated in the following way: First, any numerator term in k^2 can be written in terms of the denominator factors and the cross term $k_i \cdot k_j$ can be rewritten using

$$k_1 \cdot k_2 = \frac{1}{2} ((k_1 + k_2)^2 - k_1^2 - k_2^2) \quad (3.11)$$

Also, by using the relations

$$\frac{k^2}{k^2 + M^2} = 1 - \frac{M^2}{k^2 + M^2} \quad (3.12)$$

$$\frac{k^2 + M^2}{k^2} = 1 + \frac{M^2}{k^2} \quad (3.13)$$

the numerator is reduced to factors independent of the integration variables. The next step is to separate the denominator with the use of partial fractions. In particular,

$$\frac{1}{k^2 (k^2 + M^2)} = \frac{1}{M^2} \left(\frac{1}{k^2} - \frac{1}{k^2 + M^2} \right) \quad (3.14)$$

By separating these denominator terms, all terms will be dependent on one of two integral types. The first is

$$\int \frac{[d^D k_1] [d^D k_2]}{(k_1^2 + M_W^2)^a (k_2^2 + M_W^2)^b (k_1 + k_2)^{2c}} \quad (3.15)$$

The second type is

$$\int \frac{[d^D k_1] [d^D k_2]}{(k_1^2 + M_W^2)^a (k_1 + k_2)^{2b} k_2^{2c}} \quad (3.16)$$

Both of these types can be solved explicitly using the techniques of Appendix C.3.

The next region to account for is the region where one loop momentum is large and the other is small. To illustrate this region, take $k_1 \sim M_W$ and $k_2 \ll M_W$. With this assignment, the small parameters consist of the list given in the previous region along with k_2 . With the same momentum assignments as before, the expansions are:

$$\begin{aligned} \frac{1}{(k_1 - p)^2 + M_W^2} &= \frac{1}{k_1^2 + M_W^2} \sum_{n=0}^{\infty} \left(\frac{2k_1 \cdot p - p^2}{k_1^2 + M_W^2} \right)^n \\ \frac{1}{k_1^2 + m_i^2} &= \frac{1}{k_1^2} \sum_{n=0}^{\infty} \left(\frac{-m_i^2}{k_1^2} \right)^n \\ \frac{1}{(k_1 + k_2 - p)^2 + m_i^2} &= \frac{1}{k_1^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2k_1 \cdot (k_2 - p) + (k_2 - p)^2 + m_i^2}{k_1^2} \right)^n \\ \frac{1}{(k_2 - p)^2 + M_W^2} &= \frac{1}{M_W^2} \sum_{n=0}^{\infty} \left(\frac{-(k_2 - p)^2}{M_W^2} \right)^n \end{aligned} \quad (3.17)$$

with the propagator for m_j remaining untouched. After these expansions the integrals are now in the form

$$\int [d^D k_1] \frac{f(k_1)}{k_1^{2a} (k_1^2 + M_W^2)^b} \int [d^D k_2] \frac{f(k_2)}{(k_2^2 + m_j^2)^c} \quad (3.18)$$

which corresponds to a product of known one-loop integrals (see Appendix C.3).

The region where $k_2 \sim M_W$ and $k_1 \ll M_W$ are handled in the exact same manner.

Before moving on to the region where both momenta are small, we must look at the case when $k_1 \sim k_2 \sim M_W$ but $(k_1 + k_2) \ll M_W$. In this case, let

$(k_1 + k_2) = k_3$ and rewrite k_2 in terms of k_1 and k_3 . With this, the expansions are:

$$\begin{aligned}
 \frac{1}{(k_1 - p)^2 + M_W^2} &= \frac{1}{k_1^2 + M_W^2} \sum_{n=0}^{\infty} \left(\frac{2k_1 \cdot p - p^2}{k_1^2 + M_W^2} \right)^n \\
 \frac{1}{(k_3 - k_1 - p)^2 + M_W^2} &= \frac{1}{k_1^2 + M_W^2} \sum_{n=0}^{\infty} \left(\frac{2k_1 \cdot (k_3 - p) - (k_3 - p)^2}{k_1^2 + M_W^2} \right)^n \\
 \frac{1}{k_1^2 + m_i^2} &= \frac{1}{k_1^2} \sum_{n=0}^{\infty} \left(\frac{-m_i^2}{k_1^2} \right)^n \\
 \frac{1}{(k_3 - k_1)^2 + m_j^2} &= \frac{1}{k_1^2} \sum_{n=0}^{\infty} \left(\frac{2k_1 \cdot k_3 - k_3^2 - m_j^2}{k_1^2} \right)^n
 \end{aligned} \tag{3.19}$$

By making the momentum shift $k_3 \rightarrow k_3 + p$, the integral separates into

$$\int [d^D k_1] \frac{f(k_1)}{k_1^{2a} (k_1^2 + M_W^2)^b} \int [d^D k_3] \frac{f(k_3)}{(k_3^2 + m_i^2)^c} \tag{3.20}$$

which is easily solved.

The last region that must be dealt with is the region with $k_1 \ll M_W$ and $k_2 \ll M_W$. With this constraint, the first expansion to make is for the W -propagators:

$$\frac{1}{(k_{1,2} - p)^2 + M_W^2} = \frac{1}{M_W^2} \sum_{n=0}^{\infty} \left(\frac{-(k_{1,2} - p)^2}{M_W^2} \right)^n \tag{3.21}$$

At this point, the calculation becomes more complicated. Previously there was no worry about whether the internal charged lepton was a muon or an electron. This distinction will now come into play. Starting with the muon case, it will be demonstrated that further constraints must be put on the momentum in order to evaluate the diagrams.

First, consider the region with $k_1 \sim k_2 \sim m_\mu$. The small parameters are then p, m_e, m_i, m_j . The neutrino propagators and the muon propagator can be expanded as

$$\frac{1}{k_{1,2}^2 + m_{i,j}^2} = \frac{1}{k_{1,2}^2} \sum_{n=0}^{\infty} \left(\frac{-m_{i,j}^2}{k_{1,2}^2} \right)^n$$

$$\frac{1}{(k_1 + k_2 - p)^2 + m_\mu^2} = \frac{1}{(k_1 + k_2)^2 + m_\mu^2} \sum_{n=0}^{\infty} \left(\frac{2p \cdot (k_1 + k_2) - p^2}{(k_1 + k_2)^2 + m_\mu^2} \right)^n \quad (3.22)$$

With two shifts in momentum, $k_1 \rightarrow k_1 - k_2$ and then $k_1 \rightarrow -k_1$, the terms left depend on integrals of the form

$$\int \frac{[d^D k_1] [d^D k_2] f(k_1, k_2)}{k_1^{2a} (k_1 + k_2)^{2b} (k_1^2 + m_\mu^2)^c} \quad (3.23)$$

which can quickly be reduced to the solvable integral types described in Appendix C.3.

Moving on to the region with $k_1 \sim m_\mu$ and $k_2 \ll m_\mu$, the expansions are:

$$\begin{aligned} \frac{1}{(k_1 + k_2 - p)^2 + m_\mu^2} &= \frac{1}{k_1^2 + m_\mu^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2k_1 \cdot (k_2 - p) + (k_2 - p)^2}{k_1^2 + m_\mu^2} \right)^n \\ \frac{1}{k_1^2 + m_j^2} &= \frac{1}{k_1^2} \sum_{n=0}^{\infty} \left(\frac{-m_j^2}{k_1^2} \right)^n \end{aligned} \quad (3.24)$$

with the propagator for m_j remaining unexpanded. The terms now depend on factorized integrals of the type

$$\int \frac{[d^D k_1] [d^D k_2] f(k_1, k_2)}{k_1^{2a} (k_1^2 + m_\mu^2)^b (k_2^2 + m_j^2)^c k_2^{2d}} \quad (3.25)$$

The region with $k_1 \ll m_\mu$ and $k_2 \sim m_\mu$ is handled in the exact same manner.

Finally, there is a region which has $k_1 \sim k_2 \ll m_\mu$ so that the muon propagator can be expanded as

$$\frac{1}{(k_1 + k_2 - p)^2 + m_\mu^2} = \frac{1}{m_\mu^2} \sum_{n=0}^{\infty} \left(\frac{-(k_1 + k_2 - p)^2}{m_\mu^2} \right)^n \quad (3.26)$$

and the integrals left are of the form

$$\int \frac{[d^D k_1] [d^D k_2] f(k_1, k_2)}{k_1^{2a} (k_1^2 + m_i^2)^b (k_2^2 + m_j^2)^c k_2^{2d}} \quad (3.27)$$

which can be factored into two solvable one-loop integrals.

Moving on to the electron case, there are many similarities to the muon case. However, there is one major difference caused by the fact that the momentum

p runs through the internal charged lepton line. Since $p^2 = -m_e^2$ in Euclidean space, the mass term is cancelled in the electron propagator. After the appropriate momentum shifts, we are left with integrals of the type

$$\int \frac{[d^D k_1] [d^D k_2] f(k_1, k_2)}{k_1^{2a} (k_1 + k_2)^{2b} (k_1^2 + 2k_1 \cdot p)^c k_2^{2d}} \quad (3.28)$$

for the region with $k_1 \sim k_2 \sim m_e$ and

$$\int \frac{[d^D k_1] [d^D k_2] f(k_1, k_2)}{k_1^{2a} (k_1^2 + 2k_1 \cdot p)^b (k_2^2 + m_j^2)^c} \quad (3.29)$$

for the region $k_1 \sim m_e$ and $k_2 \ll m_e$. For the region where $k_1 \ll m_e$ and $k_2 \sim m_e$, interchange k_1 and k_2 and m_j with m_i in Eq. (3.29).

3.2.2 EDM with Neutrino Mass Comparable to M_W

In this region, the EDM of the electron will be calculated with the assumption that $m_{i,j} \sim M_W$. Looking at Eq. (3.8), it can be seen that the regions where one momentum is large and the other is small will result in a scaleless integral for the small momentum. Therefore, these regions will not contribute to the overall result. The only two regions that will contribute are when both momenta are large or when both momenta are large but their sum is small. Before the propagators are expanded, the neutrino masses will be written in terms of one heavy neutrino for the purpose of comparing to other regions later on. Thus, the neutrino masses become

$$m_i = M_\nu + \Delta \quad (3.30)$$

$$m_j = M_\nu - \Delta \quad (3.31)$$

where Δ is a small parameter compared to M_W . Furthermore, the large neutrino mass, M_ν , can be written in terms of M_W to simplify the integrals. With

$$M_\nu = M_W + \delta, \quad (3.32)$$

Eqs. (3.30) and (3.31) can be rewritten as

$$m_i = M_W + \delta + \Delta \quad (3.33)$$

$$m_j = M_W + \delta - \Delta \quad (3.34)$$

and the propagators are ready to be expanded. The first step is to expand all the kinematically small parameters. Thus, in the region where both momenta are large, the internal charged lepton propagator becomes

$$\frac{1}{(k_1 + k_2 - p)^2 + m_l^2} = \frac{1}{(k_1 + k_2)^2} \sum_{n=0}^{\infty} \left(\frac{2p \cdot (k_1 + k_2) - p^2 - m_l^2}{(k_1 + k_2)^2} \right)^n \quad (3.35)$$

while the W -propagators and the neutrino propagators remain the same.

At this point, it is advantageous to substitute our redefinition of the neutrino masses, Eqs. (3.33) and (3.34), and expand in the small parameters Δ and δ .

$$\frac{1}{k_{1,2}^2 + m_{i,j}^2} = \frac{1}{k_{1,2}^2 + M_W^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{\delta^2 + \Delta^2 + 2(M_W \delta \pm M_W \Delta \pm \Delta \delta)}{k_{1,2}^2 + M_W^2} \right)^n \quad (3.36)$$

After this expansion and an averaging over the directions of p , all terms can be written in terms of integrals of the type

$$\int \frac{[d^D k_1] [d^D k_2]}{(k_1^2 + M_W^2)^a (k_1 + k_2)^{2b} (k_2^2 + M_W^2)^c} \quad (3.37)$$

This class of two-loop integrals can be solved explicitly, as shown in Appendix C.3.

In the region where the two momenta are large but their sum, $k_1 + k_2 = k_3$ is small, all propagators involving k_2 must be treated as mentioned in Section 3.2.1 where the neutrino masses were small. All terms remaining now depend on integrals of the type

$$\int [d^D k_1] [d^D k_3] \frac{f(k_1, k_3)}{k_1^{2a} (k_1^2 + M_W^2)^b (k_3^2 + m_l^2)^c k_3^{2d}} \quad (3.38)$$

which can be evaluated as a sequence of one-loop integrals.

3.2.3 EDM with Neutrino Mass Larger than M_W

In this limit of the electric dipole moment of the electron, the mass of the Majorana particle is taken to be much larger than that of M_W . Once again, any momentum region where k_1 and $k_2 \ll M_W$ will not contribute as in the previous case. Thus, the same regions must be considered as in the limit with $M_\nu \sim M_W$. However, in this case, there is an added complication since the neutrino mass and M_W are no longer on the same scale. Thus, the momentum regions must be further divided depending on whether they are of the scale of the neutrino mass or M_W . The same conventions are used as in Eqs. (3.30) and (3.31).

The first region is concerned with $k_1 \sim k_2 \sim M_\nu \gg M_W$. The momenta can be rerouted and the propagators can be expanded as (only terms at most linear in Δ were retained)

$$\begin{aligned}
 \frac{1}{k_1^2 + M_W^2} &= \frac{1}{k_1^2} \sum_{n=0}^{\infty} \left(\frac{-M_W^2}{k_1^2} \right)^n \\
 \frac{1}{(k_1 + p)^2 + m_i^2} &= \frac{1}{k_1^2 + M_\nu^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{p^2 + 2k_1 \cdot p + 2M_\nu \Delta}{k_1^2 + M_\nu^2} \right)^n \\
 \frac{1}{(k_1 + k_2)^2 + m_i^2} &= \frac{1}{(k_1 + k_2)^2} \sum_{n=0}^{\infty} \left(\frac{-m_i^2}{(k_1 + k_2)^2} \right)^n \\
 \frac{1}{(k_2 - p)^2 + M_W^2} &= \frac{1}{k_2^2} \sum_{n=0}^{\infty} \left(\frac{2k_2 \cdot p - p^2 - M_W^2}{k_2^2} \right)^n
 \end{aligned} \tag{3.39}$$

As in Section 3.2.1, partial fractions are used to obtain terms that are dependent on integrals of the form

$$\int \frac{[d^D k_1] [d^D k_2]}{(k_1^2 + M_\nu^2)^a (k_2^2 + M_\nu^2)^b (k_1 + k_2)^{2c}} \tag{3.40}$$

or

$$\int \frac{[d^D k_1] [d^D k_2]}{(k_1^2 + M_\nu^2)^a (k_1 + k_2)^{2b} k_2^{2c}} \tag{3.41}$$

The next region deals with both momenta on the order of the M_W . In this region, the $W(k_1)$ -propagator remains the same, but the other propagators can

be expanded as

$$\begin{aligned}
 \frac{1}{(k_1 + p)^2 + m_i^2} &= \frac{1}{M_\nu^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{(k_1 + p)^2 + 2M_\nu \Delta}{M_\nu^2} \right)^n \\
 \frac{1}{(k_1 + k_2)^2 + m_l^2} &= \frac{1}{(k_1 + k_2)^2} \sum_{n=0}^{\infty} \left(\frac{-m_l^2}{(k_1 + k_2)^2} \right)^n \\
 \frac{1}{k_2^2 + m_j^2} &= \frac{1}{M_\nu^2} \sum_{n=0}^{\infty} \left(\frac{2M_\nu \Delta - k_2^2}{M_\nu^2} \right)^n \\
 \frac{1}{(k_2 - p)^2 + M_W^2} &= \frac{1}{k_2^2 + M_W^2} \sum_{n=0}^{\infty} \left(\frac{2k_2 \cdot p - p^2}{k_2^2 + M_W^2} \right)^n
 \end{aligned} \tag{3.42}$$

leaving terms that depend on integrals of the type

$$\int \frac{[d^D k_1] [d^D k_2] f(k_1, k_2)}{(k_1^2 + M_W^2)^a (k_1 + k_2)^{2b} (k_2^2 + M_W^2)^c} \tag{3.43}$$

In the third region, one momentum is on the order of the neutrino mass and the other is on the order of the M_W . Again, to illustrate this region, consider $k_1 \sim M_\nu$ and $k_2 \sim M_W$. The propagators in this region can be expanded as

$$\begin{aligned}
 \frac{1}{k_1^2 + M_W^2} &= \frac{1}{k_1^2} \sum_{n=0}^{\infty} \left(\frac{-M_W^2}{k_1^2} \right)^n \\
 \frac{1}{(k_1 + p)^2 + m_i^2} &= \frac{1}{k_1^2 + M_\nu^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{p^2 + 2k_1 \cdot p + 2M_\nu \Delta}{k_1^2 + M_\nu^2} \right)^n \\
 \frac{1}{(k_1 + k_2)^2 + m_l^2} &= \frac{1}{k_1^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{k_2^2 + 2k_2 \cdot k_1 + m_l^2}{k_1^2} \right)^n \\
 \frac{1}{k_2^2 + m_j^2} &= \frac{1}{M_\nu^2} \sum_{n=0}^{\infty} \left(\frac{2M_\nu \Delta - k_2^2}{M_\nu^2} \right)^n \\
 \frac{1}{(k_2 - p)^2 + M_W^2} &= \frac{1}{k_2^2 + M_W^2} \sum_{n=0}^{\infty} \left(\frac{2k_2 \cdot p - p^2}{k_2^2 + M_W^2} \right)^n
 \end{aligned} \tag{3.44}$$

leaving terms that depend on integrals of the form

$$\int \frac{[d^D k_1] [d^D k_2] f(k_1, k_2)}{k_1^{2a} (k_1^2 + M_\nu^2)^b (k_2^2 + M_W^2)^c} \tag{3.45}$$

The next region that must be dealt with is where both momenta are large but their sum is small. Again, there is some complication and the region must be

split up into two subregions. The first subregion sees $k_1 \sim k_2 \sim M_\nu$ and $k_1 + k_2 \equiv k_3 \sim m_l$. The expansion of the propagators in this region are as follows

$$\begin{aligned} \frac{1}{k_1^2 + M_W^2} &= \frac{1}{k_1^2} \sum_{n=0}^{\infty} \left(\frac{-M_W^2}{k_1^2} \right)^n \\ \frac{1}{(k_1 + p)^2 + m_i^2} &= \frac{1}{k_1^2 + M_\nu^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{p^2 + 2k_1 \cdot p + 2M_\nu \Delta}{k_1^2 + M_\nu^2} \right)^n \\ \frac{1}{k_2^2 + m_j^2} &= \frac{1}{k_1^2 + M_\nu^2} \sum_{n=0}^{\infty} \left(\frac{2M_\nu \Delta + 2k_3 \cdot k_1 - k_3^2}{k_1^2 + M_\nu^2} \right)^n \\ \frac{1}{(k_2 - p)^2 + M_W^2} &= \frac{1}{k_1^2} \sum_{n=0}^{\infty} \left(\frac{2k_1 \cdot (k_3 - p) - (k_3 - p)^2 - M_W^2}{k_1^2} \right)^n \end{aligned} \quad (3.46)$$

which leaves terms that are dependent on integrals of the form

$$\int \frac{[d^D k_1] [d^D k_3] f(k_1, k_3)}{k_1^{2a} (k_1^2 + M_\nu^2)^b (k_3^2 + m_l^2)^c} \quad (3.47)$$

The last region corresponds to $k_1 \sim k_2 \sim M_W$ and $k_1 + k_2 \equiv k_3 \sim m_l$. The propagators can be expanded as

$$\begin{aligned} \frac{1}{(k_1 + p)^2 + m_i^2} &= \frac{1}{M_\nu^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{(k_1 + p)^2 + 2M_\nu \Delta}{M_\nu^2} \right)^n \\ \frac{1}{k_2^2 + m_j^2} &= \frac{1}{M_\nu^2} \sum_{n=0}^{\infty} \left(\frac{2M_\nu \Delta - (k_3 - k_1)^2}{M_\nu^2} \right)^n \\ \frac{1}{(k_2 - p)^2 + M_W^2} &= \frac{1}{k_1^2 + M_W^2} \sum_{n=0}^{\infty} \left(\frac{2k_1 \cdot (k_3 - p) - (k_3 - p)^2}{k_1^2 + M_W^2} \right)^n \end{aligned} \quad (3.48)$$

which leaves terms that depend on integrals of the form

$$\int \frac{[d^D k_1] [d^D k_3] f(k_1, k_3)}{(k_1^2 + M_W^2)^a (k_3^2 + m_l^2)^b} \quad (3.49)$$

which are readily solved.

Chapter 4

Results and Discussion

Before the analytic results are discussed, it is necessary to understand how the sum over the mixing matrix takes place and how the CP-violating phase affects the calculation.

As shown in Section 2.2.6, the mixing between two Majorana neutrino flavours will take the form

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} c & se^{i\delta} \\ -se^{-i\delta} & c \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad (4.1)$$

where $e^{i\delta}$ is the CP-violating phase factor.

The flavour mixing will effect the couplings and will enter the amplitude in the form

$$F_l = \sum_{i,j} U_{ej}^* U_{lj}^* U_{li} U_{ei} a(m_i, m_j) A(l) \quad (4.2)$$

where $a(m_i, m_j)$ is a function of the Majorana neutrino masses and $A(l)$ is the component of the amplitude which depends on the internal charged lepton masses.

In the case that $l = \mu$, the flavour mixing factor becomes

$$F_\mu = \sum_{i,j=1}^2 U_{ej}^* U_{\mu j}^* U_{\mu i} U_{ei} a(m_i, m_j) A(l) \quad (4.3)$$

$$= c^2 s^2 \{a(m_1, m_1) + a(m_2, m_2) + 2i \sin(2\delta) a(m_2, m_1)\} A(\mu)$$

For the case that $l = e$, the flavour mixing factor becomes

$$\begin{aligned} F_e &= \sum_{i,j=1}^2 U_{ej}^* U_{ej} U_{ei} U_{ei} a(m_i, m_j) A(e) \\ &= \{c^4 a(m_1, m_1) - 2is^2 c^2 \sin(2\delta) a(m_2, m_1) + s^4 a(m_2, m_2)\} A(e) \end{aligned} \quad (4.4)$$

It is seen here that the internal charged lepton contributions come in with different signs and that if there is no dependence on the mass on the internal charged lepton, the contribution disappears. The results are also in the form of anti-symmetric functions in the Majorana neutrino masses, analagous to the GIM mechanism in the quark sector. Thus, any degeneracy in the Majorana masses will cause the EDM to vanish. With these properties, the final result for the EDM of the electron in the limit that the Majorana neutrino masses are much smaller than that of the M_W is

$$\begin{aligned} d_e|_{m_{i,j} \ll M_W} &= s^2 c^2 \sin(2\delta) \frac{em_e m_i m_j G_F^2}{8\pi^4 M_W^2} \left\{ \left(\frac{169}{108} + \frac{\pi^2}{18} - \frac{1}{3} \ln \frac{m_e^2}{m_\mu^2} \right) (m_i^2 - m_j^2) \right. \\ &\quad + \frac{11}{18} \left[m_i^2 \ln \frac{m_i^2}{m_e^2} - m_j^2 \ln \frac{m_j^2}{m_e^2} \right] \\ &\quad \left. - \frac{1}{6} \ln \frac{m_e^2}{m_\mu^2} \left[m_i^2 \ln \frac{m_i^4}{m_e^2 m_\mu^2} - m_j^2 \ln \frac{m_j^4}{m_e^2 m_\mu^2} \right] \right\} \end{aligned} \quad (4.5)$$

where terms suppressed by factors of $(m_e/m_\mu)^2$, $(m_e/M_W)^2$ and $(m_\mu/M_W)^2$ have been neglected. The mass dependence of the EDM in Eq. (4.5) agrees with the result of [12]. However, with the use asymptotic expansions, an analytic result has been obtained which was previously thought to be unattainable. An estimate for the EDM can now be made. Using $m_{i,j} \sim 1\text{eV}$ and $m_i^2 - m_j^2 = 10^{-2}\text{eV}^2$, the EDM for the electron is

$$d_e \sim 10^{-55} \text{e cm} \quad (4.6)$$

which is comparable to Standard Model results and is well below any experimental limit in the foreseeable future. The discrepancy between the results in Eq. (4.6) and [12] relies on the fact that it is unclear what values for the neutrino mass should be used. By forcing the neutrino mass to be as high as the order of the electron mass, one could obtain an EDM on the order of 10^{-35} e cm, which could soon be within experimental reach. With the more conservative result, though, it is seen that the EDM is not the best method for searching for CP violation in the lepton sector.

The use of asymptotic expansions to obtain analytic solutions allows for a calculation to find the behaviour of the EDM as the mass of the Majorana particles increase. For the scenario where the Majorana particle is approximately that of the M_W the neutrino masses have been set to $m_{i,j} \rightarrow M_{i,j} = M_\nu \pm \Delta$ and $M_\nu = M_W + \delta$ as discussed in Section 3.2.2. With the properties of the mixing matrix given in Eqs. (4.4) and (4.5), the result is

$$d_e|_{m_{i,j} \sim M_W} = s^2 c^2 \sin(2\delta) \frac{em_e m_\mu^2 G_F^2 \Delta}{8\pi^4 M_W} \left(\frac{M_{D_i}}{M_\nu}\right)^2 \left(\frac{M_{D_j}}{M_\nu}\right)^2 f(x) \quad (4.7)$$

where

$$\begin{aligned} f(x) \simeq & -\frac{1957}{21600} + \frac{1}{20} \ln \frac{m_\mu^2}{M_W^2} \\ & + \left(\frac{1-x}{x}\right) \left[\frac{-4657}{151200} + \frac{1}{60} \ln \frac{m_\mu^2}{M_W^2} \right] \\ & + \left(\frac{1-x}{x}\right)^2 \left[\frac{13777}{1058400} + \frac{13}{420} \ln \frac{m_\mu^2}{M_W^2} \right] \\ & + \left(\frac{1-x}{x}\right)^3 \left[\frac{2341}{352800} + \frac{3}{140} \ln \frac{m_\mu^2}{M_W^2} \right] \end{aligned} \quad (4.8)$$

Here, $x = M_W/M_\nu$ and the factors of M_D/M_ν come from the weak coupling of the heavy Majorana state given in Eq. (2.73). It is seen that the antisymmetry of the Majorana masses is accounted for in Δ and that any degeneracy in the neutrino mass causes Δ , and therefore d_e , to vanish.

The final result for the EDM of the electron is in the scenario where the Majorana neutrino is much heavier than the mass of the W . In this case, the neutrino masses have been set to $m_{i,j} = M_\nu \pm \Delta$, so that

$$d_e|_{m_{i,j} \gg M_W} = s^2 c^2 \sin(2\delta) \frac{em_e m_\mu^2 G_F^2 \Delta}{8\pi^4 M_W} \left(\frac{M_{D_i}}{M_\nu}\right)^2 \left(\frac{M_{D_j}}{M_\nu}\right)^2 g(x) \quad (4.9)$$

where

$$\begin{aligned} g(x) \simeq & x \left[\frac{161}{36} - \frac{4\pi^2}{9} + \frac{1}{6} \ln \frac{m_\mu^2}{M_W^2} + \frac{1}{3} \ln x \right] \\ & + x^3 \left[\frac{82}{3} - \frac{22\pi^2}{9} - \frac{2}{3} \ln \frac{m_\mu^2}{M_W^2} + \frac{5}{3} \ln x \right] \\ & + x^5 \left[\frac{3787}{36} - \frac{77\pi^2}{12} - \frac{13}{2} \ln \frac{m_\mu^2}{M_W^2} + 31 \ln x \right. \\ & \quad \left. - 6 \ln \frac{m_\mu^2}{M_W^2} \ln x - 3 \ln^2 x \right] \end{aligned} \quad (4.10)$$

By expanding the results of Eqs. (4.8) and (4.10) to sufficiently high orders ($f(x)$ and $g(x)$ and been expanded to 16 and 9 terms, respectively), the EDM as a function of the mass of the Majorana neutrino is obtained, as illustrated in Figure 4.1. Note that the EDM of the electron vanishes, as expected, as the Majorana neutrino becomes infinitely heavy.

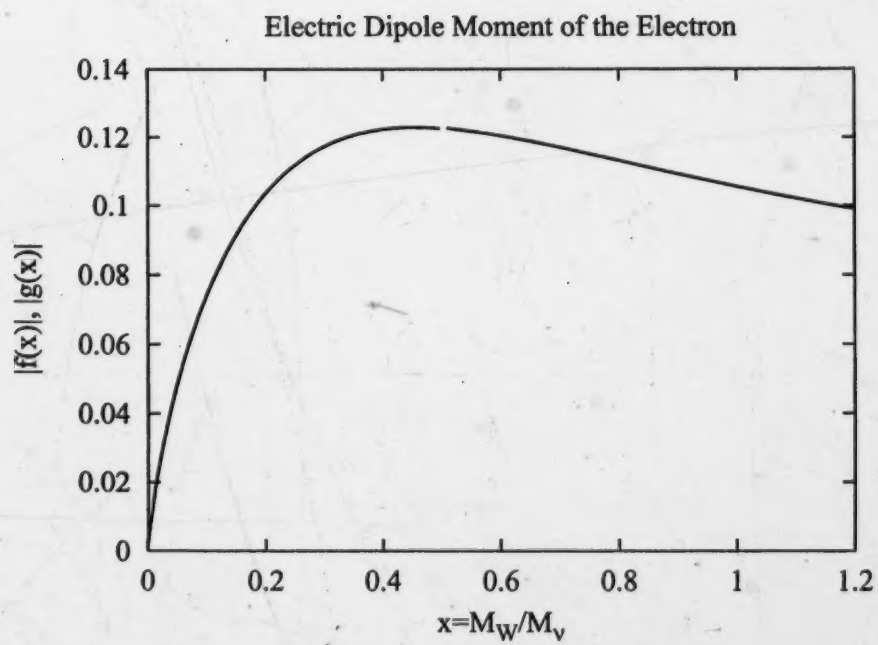


Figure 4.1: EDM of the electron in the limit of a heavy Majorana particle

Chapter 5

Conclusions

This thesis has investigated the effects of Majorana neutrinos on the EDM of the electron.

By introducing the discrete transformations of charge conjugation, parity and time reversal, it has been shown that a non-zero EDM for a fundamental fermion violates CP invariance. CP violation has been explained in the quark sector, where using the kaon system as an example, it was illustrated that CP violation could be parametrised by a complex phase in the CKM quark mixing matrix.

To extend the idea of CP violation into the lepton sector, massive neutrinos were introduced. Massive neutrinos allow for the possibility of neutrino mixing and the violation of lepton flavour. They also give rise to a mixing matrix analogous to the CKM matrix. Along with neutrino mixing, a neutrino mass allows the possibility that the neutrino is a Majorana particle, meaning that the particle and anti-particle are the same apart from a phase factor. This phase factor is responsible for decreasing the number of generations required to introduce a complex phase in the mixing matrix. Only two lepton generations are needed for a CP-violating phase as compared to three generations in the

quark sector.

By considering a left-right symmetric model for the electroweak interaction, it was demonstrated that two Majorana states are produced for each generation of lepton. Through the seesaw mechanism, one state obtains a mass, M_R , that is on the order of the right-handed symmetry breaking scale, which would have to be above M_W as ruled by experiment. The lighter state acquires a mass on the order of M_D^2/M_R , where M_D is a typical value of a Dirac mass. These states allow for the calculation of the EDM in the scenario where the neutrino mass is much smaller than M_W or when the Majorana mass increases beyond M_W .

To calculate the EDM, an operator was developed that projected out the appropriate piece from the amplitudes of the Feynman diagrams in order to simplify the calculation into that of a trace. The diagrams were then calculated using the method of asymptotic expansions. Three distinct calculational scenarios were explored in detail to outline the method of asymptotic expansions.

The result for $m_{i,j} \ll M_W$ has been found to be in agreement with previous attempts, however, in this work, an analytic result has been found. The result indicates that an EDM for the electron can be induced at the two-loop level if Majorana particles are considered. The magnitude of the EDM has been found to be

$$d_e = 10^{-55} e \text{ cm} \quad (5.1)$$

which will not be observable by any experiment in the near future.

As the Majorana mass increases, the electron EDM is shown to behave as expected. In particular, the EDM vanishes as the Majorana mass becomes very large and has a non-zero value as the mass decreases.

This project has also built up calculational tools that will be useful in order to investigate the EDM of the electron further. For example, further study can be made in the situation where one of the Majorana particles is that of the heavy

state and the other is the light state. This would provide a generalisation of this work and it may lead to an electron EDM that is less severely suppressed.

Appendix A

Notations and Conventions

The notations and conventions used follow that of [30]. In particular, $\hbar = c = 1$ so that masses, energies and inverse lengths are measured in units of eV.

The spinors $u^s(p)$ and $v^s(p)$ satisfy the Dirac equation in the form

$$(\not{p} - m)u^s(p) = (\not{p} + m)v^s(p) = \bar{u}^s(p)(\not{p} - m) = \bar{v}^s(p)(\not{p} + m) = 0 \quad (\text{A.1})$$

where $\not{p} \equiv \gamma^\mu p_\mu$. The Dirac matrices obey the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad (\text{A.2})$$

with the metric

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{A.3})$$

When an explicit basis is needed for calculations or identities, the chiral basis is used. The γ -matrices, in this basis, take the form

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{A.4})$$

where σ^i are the standard Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.5})$$

Spinors that satisfy Eq. (A.1), are

$$u^{1/2}(p) = \begin{pmatrix} 1 \\ 0 \\ (E + p_z)/m \\ (p_x + ip_y)/m \end{pmatrix} \quad u^{-1/2}(p) = \begin{pmatrix} 0 \\ 1 \\ (p_x - ip_y)/m \\ (E - p_z)/m \end{pmatrix} \quad (\text{A.6})$$

$$v^{1/2}(p) = \begin{pmatrix} (p_x - ip_y)/m \\ -(E + p_z)/m \\ 0 \\ 1 \end{pmatrix} \quad v^{-1/2}(p) = \begin{pmatrix} -(E - p_z)/m \\ -(p_x + ip_y)/m \\ 1 \\ 0 \end{pmatrix} \quad (\text{A.7})$$

In the representation given in Eq. (A.4), the charge conjugation matrix has the form

$$C = i\gamma^0\gamma^2 \quad (\text{A.8})$$

which is easily seen to have the following properties:

$$C^{-1} = -C = C^\dagger = C^T = -C^* \quad (\text{A.9})$$

and

$$\{C, \gamma^0\} = 0 \quad (\text{A.10})$$

$$[C, \gamma^5] = 0 \quad (\text{A.11})$$

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^T \quad (\text{A.12})$$

The charge conjugated wavefunction satisfies

$$\psi^c = C\bar{\psi}^T \quad (\text{A.13})$$

$$\bar{\psi}^c = \psi^T C = \psi^T C^{-1} \quad (\text{A.14})$$

The spinors given in Eqs. (A.6) and (A.7) satisfy the relations

$$u^s(p) = -i\gamma^2 (v^s(p))^* \quad v^s(p) = -i\gamma^2 (u^s(p))^* \quad (\text{A.15})$$

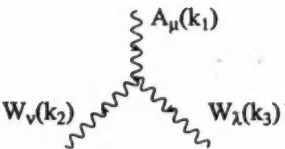
The electromagnetic field tensor takes the form

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (\text{A.16})$$

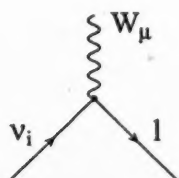
Appendix B

Feynman Rules

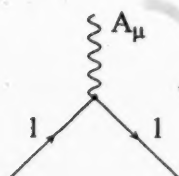
B.1 Vertices



$$= -ie [(k_1 - k_2)_\lambda g_{\mu\nu} + (k_2 - k_3)_\mu g_{\nu\lambda} + (k_3 - k_1)_\nu g_{\lambda\mu}]$$



$$= \frac{ig_w}{2\sqrt{2}} \gamma_\mu (1 - \gamma_5) U_{il}$$



$$= ie\gamma_\mu$$

B.2 Propagators

$$\begin{array}{c} \text{---} \longrightarrow \\ p \end{array} = \frac{i(\not{p} + m)}{p^2 - m^2}$$

$$\begin{array}{c} \mu \text{ --- } \text{wavy line} \text{ --- } \nu \\ k \end{array} = \frac{-i}{k^2 - M_W^2} \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{M_W^2} \right]$$

Appendix C

Loop Integrals

C.1 Motivation

Feynman diagrams are a calculational tool used to describe particle interactions. Using the Feynman rules, it is possible to construct Feynman diagrams that contain closed internal loops within which undetermined momenta flow. These loop momenta must be integrated over all values.

The most basic loop integral that can arise is

$$\int \frac{[d^4k]}{(k^2 + m^2)^n} \quad (\text{C.1})$$

where $[d^4k] \equiv d^4k/(2\pi)^4$, k is the loop momentum, m is an external parameter with the dimensions of mass and n is an integer.

If $n < 2$, the integral will diverge due to the effects of large values of k . In the context of a physically meaningful calculation, an assortment of loop integral divergences will cancel in order to produce a finite physical result. Nevertheless, it is important to be able to cope with individual divergences so that a calculation can be performed in manageable pieces. Regularisation is the procedure by which these divergences are managed so as not to lose track of the contributions to the eventual finite result. In this thesis, a process known as dimensional

regularisation is used. In dimensional regularisation, the number of space-time dimensions, D , is left unspecified and the calculation is performed. The convention used here sets $D = 4 - 2\epsilon$, so that when $\epsilon \rightarrow 0$ at the end of the calculation, the familiar $D = 4$ limit is recovered. The power of this method lies in the fact that the divergences become isolated as pole terms in a Laurent series. Once all the calculations for an interaction have been completed, the pole terms will cancel. Using dimensional regularisation, the result of the integral in Eq. (C.1) is [31]

$$\int \frac{[d^D k]}{(k^2 + m^2)^n} = \frac{\Gamma(n - 2 + \epsilon)}{(4\pi)^{2-\epsilon} (m^2)^{n-2+\epsilon} \Gamma(n)} \quad (\text{C.2})$$

C.2 Techniques

One method of solving common loop integrals will be illustrated using a type of integral that was encountered in the calculation of the electron EDM:

$$\int \frac{[d^D k] f(k)}{(k^2 + m^2)^a (k^2)^b} \quad (\text{C.3})$$

A Feynman parameter, x , can be introduced via the identity [30]

$$\frac{1}{A_1^{m_1} A_2^{m_2}} = \frac{\Gamma(m_1 + m_2)}{\Gamma(m_1) \Gamma(m_2)} \int_0^1 dx \frac{x^{m_1-1} (1-x)^{m_2-1}}{[xA_1 + (1-x)A_2]^{m_1+m_2}} \quad (\text{C.4})$$

so that the loop integral becomes

$$\frac{\Gamma(a+b)}{(a-1)! \Gamma(b)} \int_0^1 dx x^{a-1} (1-x)^{b-1} \int \frac{[d^D k] f(k)}{(k^2 + xm^2)^{a+b}} \quad (\text{C.5})$$

The denominator of the momentum integral now takes the form of the denominator in Eq. (C.1). In general, a shift of the loop momentum will be required in order to obtain the desired form.

To handle the numerator factors $f(k)$, notice that any odd function of k will cause the integral to vanish due to the antisymmetry of the integrand. Generalising this observation, averaging over the directions of k leaves all the k -dependence

in the numerator as $f(k^2)$, which can be absorbed into the denominator using the identity

$$\frac{k^2}{(k^2 + xm^2)^n} = \frac{1}{(k^2 + xm^2)^{n-1}} - \frac{xm^2}{(k^2 + xm^2)^n} \quad (\text{C.6})$$

If $f(k) = 1$, the k - and x -integrals can be evaluated explicitly, yielding a final result of

$$\frac{(m^2)^{2-a-b-\epsilon}}{(a-1)!} \frac{\Gamma(a+b-2+\epsilon)\Gamma(2-b-\epsilon)}{\Gamma(2-\epsilon)\Gamma(1+\epsilon)} \quad (\text{C.7})$$

The Γ functions can then be expanded as a Laurent series in ϵ in order to isolate the divergences.

C.3 List of Integrals Encountered

Four nontrivial types of loop integrals were encountered over the course of the calculation of the electron EDM. They can all be evaluated using techniques similar to those of Section C.2. For completeness, these loop integrals are classified below along with their names.

- Scaleless

$$\int \frac{[d^D k]}{k^{2a}} = 0 \quad (\text{C.8})$$

- gm2

$$\int \frac{[d^D k]}{(k^2 + m^2)^a (k^2)^{2b}} \quad (\text{C.9})$$

- one

$$\int \frac{[d^D k]}{(k+p)^{2a} k^{2b}} \quad (\text{C.10})$$

- onshell

$$\int \frac{[d^D k]}{(k^2 + 2k \cdot p)^a k^{2b}} \quad (\text{C.11})$$

- gm3

$$\int \frac{[d^D k_1] [d^D k_2]}{(k_1^2 + m^2)^a (k_2^2 + m^2)^b (k_1 + k_2)^{2c}} \quad (\text{C.12})$$

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